An Introduction to p-adic Analysis

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Introduction

In real analysis, we are used to utilising the usual absolute value and the idea of the real numbers being the completion for the set of rational numbers. This essay will introduce the idea of using p-adic numbers and the p-adic absolute value to complete the rational numbers instead. In particular, we will explore the properties of the p-adic absolute value, and results relating to it. In this essay, p is always a prime number.

1 Absolute values

1.1 Introducing the p-adic absolute value

We will begin by defining a non-archimedean absolute value in the same way Gouvêa defines it.[1]

Definition 1.1. *A* non-archimedean absolute value *on a field* $\mathbb{F}$ *is a function*

$$|\cdot| : \mathbb{F} \to \mathbb{R}_{\geq 0}$$

that satisfies the following conditions:

1. $|x| = 0 \iff x = 0$
2. $|xy| = |x||y|$ *for all* $x, y \in \mathbb{F}$
3. $|x + y| \leq |x| + |y|$ *for all* $x, y \in \mathbb{F}$
4. $|x + y| \leq \max\{|x|, |y|\}$ *for all* $x, y \in \mathbb{F}$

Note. If the fourth condition holds, this implies that the third condition also does, since $\max\{|x|, |y|\} \leq |x| + |y|$.

Definition 1.2. *If a function* $|\cdot| : \mathbb{F} \to \mathbb{R}_{\geq 0}$ *satisfies the first three conditions above but does not satisfy the fourth, it is called an* archimedean absolute value.

Example. The usual absolute value, defined by

$$|x|_\infty = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

is an archimedean absolute value. Let $x = y = 1$. Then

$$|x + y| = 2 > 1 = \max\{|x|, |y|\}$$

which does not satisfy the fourth condition.

The fourth condition is one of the most important properties of a non-archimedean absolute value. From this, we can derive most of the properties that make mathematics using p-adic structures different. We can also somewhat strengthen it, given some extra conditions.

[1] Unless specified, all definitions, theorems and lemmas are adapted from Gouvêa's text [1, §§1-4]

Proposition 1.1. *For all $x, y \in \mathbb{F}$ with $|x| \neq |y|$,*

$$|x + y| = \max\{|x|, |y|\}$$

A proof of this is given in Gouvêa's book. [1, p31]. To define the p-adic absolute value, first we must define the p-adic valuation of a number.

Definition 1.3. *Fix a prime number $p \in \mathbb{Z}$. The* p-adic valuation *on $\mathbb{Z}^\times$ is the function $v_p : \mathbb{Z}^\times \to \mathbb{R}$ defined as*

$$v_p(x) = \max\{r : p^r | x\}$$

This function can be extended to $\mathbb{Q}$ as follows; if $x = \frac{a}{b} \in \mathbb{Q}^\times$, then

$$v_p(x) = v_p(a) - v_p(b)$$

It is convenient to define $v_p(0) = \infty$.

Note. For a field $\mathbb{F}$, $\mathbb{F}^\times = \mathbb{F} \setminus \{0\}$.

The following lemma outlines some basic properties of the p-adic valuation.

Lemma 1.2. *For all $x, y \in \mathbb{Q}$,*

1. $v_p(xy) = v_p(x) + v_p(y)$
2. $v_p(x + y) \geq \min\{v_p(x), v_p(y)\}$

Proof. When either x, y, or both, are zero, the equalities are very simple to prove. This proof will be for non-zero x, y. It is adapted from the partial proof given by Gouvêa [1, p225].

1. Consider $s, t \in \mathbb{Z}^\times$. We can write $s = p^{v_p(s)}s'$, and $t = p^{v_p(t)}t'$, where neither s' nor t' are divisible by p. Then, $st = p^{v_p(s)+v_p(t)}s't'$. Since $s't'$ is not divisible by p,
$$v_p(st) = v_p(s) + v_p(t)$$
proving the equality holds for integers.
Now consider $x, y \in \mathbb{Q}^\times$. We can write $x = \frac{a}{b}, y = \frac{c}{d}$, where $a, b, c, d \in \mathbb{Z}^\times$. So, $xy = \frac{ac}{bd}$. Therefore
$$\begin{aligned} v_p(xy) = v_p\left(\frac{ac}{bd}\right) = v_p(ac) - v_p(bd) &= v_p(a) - v_p(b) + v_p(c) - v_p(d) \\ &= v_p(x) + v_p(y) \end{aligned}$$
as required.

2. Consider s, t as in part 1. Without loss of generality, let $v_p(s) \leq v_p(t)$. So, $s + t = p^{v_p(s)}(s' + p^{v_p(t)-v_p(s)}t')$. Thus
$$v_p(s + t) \geq v_p(s) = \min\{v_p(s), v_p(t)\}$$
proving the inequality holds for integers.
Now consider x, y as in part 1. Without loss of generality, let $v_p(x) \leq v_p(y)$. This implies that $v_p(ad) \leq v_p(bc)$. Therefore
$$\begin{aligned} v_p(x + y) = v_p\left(\frac{ad + bc}{bd}\right) &\geq \min\{v_p(ad), v_p(bc)\} - v_p(b) - v_p(d) \\ &= v_p(ad) - v_p(b) - v_p(d) \\ &= v_p(x) = \min\{v_p(x), v_p(y)\} \end{aligned}$$
as required. □

Definition 1.4. *For any $x \in \mathbb{Q}$, we define the* p-adic absolute value *to be*

$$|x|_p = p^{-v_p(x)}$$

with $|0|_p = 0$ to match our convention of using $v_p(0) = \infty$.

Proposition 1.3. *$|\cdot|_p$ is a non-archimedian absolute value on $\mathbb{Q}$.*

Proof. Using Lemma 1.2, it is clear the function satisfies Conditions 1 and 2 of Definition 1.1. Here, we will only prove Condition 4, since if that holds, then so does Condition 3. Let $|x| \leq |y|$. Then $v_p(y) \leq v_p(x)$. Therefore

$$|x+y| = p^{-v_p(x+y)} \leq p^{-\min\{v_p(x), v_p(y)\}} = p^{-v_p(y)} = |y| = \max\{|x|, |y|\}$$

□

Example. Using the p-adic absolute value, $|p^n|_p = \frac{1}{p^n}$. So

$$\lim_{n \to \infty} |p^n|_p = 0$$

This means that large positive powers of p are "small" with respect to $|\cdot|_p$. This contrasts with $|\cdot|_\infty$ where large negative powers of any positive integer are "small".

1.2 p-adic geometry

By setting $d_p(x, y) = |x - y|_p$, and using the properties of non-archimedian absolute values, we can see that d_p is a metric. Then $(\mathbb{Q}, d_p)$ is a metric space.

Now, we are going to explore some of the properties of ultrametric geometry. Note that all of the results of ultrametric geometry also hold in p-adic geometry.

Definition 1.5. *A metric space (X, d) is an* ultrametric space[2] *if the metric d satisfies the strong triangle inequality:*

$$d(x, z) \leq \max\{d(x, y), d(y, z)\} \qquad \forall\ x, y, z \in X$$

Proposition 1.4. *Define a metric space $(\mathbb{F}, d)$ where $d(x, y) = |x - y|$ and $|\cdot|$ is an absolute value on a field $\mathbb{F}$. Then $|\cdot|$ is non-archimedian iff the strong triangle inequality holds.*

Proof.
($\Rightarrow$) Taking the equality $x - z = (x - y) + (y - z)$ and applying the non-archimedian properties of $|\cdot|$ gives

$$|x - z| = |(x - y) + (y - z)| \leq \max\{|x - y|, |y - z|\}$$

Therefore, $d(x-z) \leq \max\{d(x-y), d(y-z)\}$, showing that it satisfies the strong triangle inequality.
($\Leftarrow$) Let $z = -k, y = 0$. Then

$$d(x, -k) \leq \max\{d(x, 0), d(0, -k)\}$$

Therefore, $|x + k| \leq \max\{|x|, |k|\}$ as required. □

[2] This definition is from Schikhof [2, pp46-47]

Since the p-adic absolute value is non-archimedian, the strong triangle inequality will hold. Therefore, the space $(\mathbb{Q}, d_p)$ is an ultrametric space.

We can now prove one of the surprising results from the properties of ultrametric spaces:

Proposition 1.5. *In an ultrametric space, all triangles are isosceles.*

Proof. Let x, y, z be three points in an ultrametric space. Then

$$\begin{aligned} d(x,y) &= |x-y| \\ d(y,z) &= |y-z| \\ d(x,z) &= |x-z| \end{aligned}$$

If, after relabelling, $d(x,y) = d(y,z)$, then the triangle is obviously isosceles. Now consider the case where $d(x,y) \neq d(y,z)$. Since

$$(x-y) + (y-z) = (x-z)$$

you can use Proposition 1.1 to show that

$$d(x,z) = \max\{d(x,y), d(y,z)\}$$

Therefore, in either case, the triangle is isosceles. □

Using the strong triangle inequality, we will prove some results about open balls. Recall the following definition:

Definition 1.6. *In a metric space (X,d), let $a \in X$, $r \in (0,\infty)$. Then the* open ball *of radius r with centre a is the set*

$$B(a,r) = \{x \in X : d(a,x) < r\}$$

Proposition 1.6. *Let $\mathbb{F}$ be a field with a non-archimedian absolute value. Then:*

1. *Every point that is contained in an open ball is a centre of that ball.*
2. *Any two open balls are either disjoint or contained in one another.*
3. *The diameter of a ball is less than or equal to its radius.*[3]

Proof. Schikhof offers proofs for some of these statements [2, p48], but this proof will be based on the more accessible ones given by Gouvêa and Robert - [1, p34] and [3, p70].

1. Consider an open ball, $B(a,r)$. Then

   $$b \in B(a,r) \iff d(b,a) < r$$

   Take any $b, x \in B(a,r)$. Then $d(x,a) < r$. So, using the strong triangle inequality

   $$d(x,b) \leq \max\{d(x,a), d(b,a)\} < r$$

   In other words, $x \in B(b,r)$. Since this holds for all x, $B(a,r) = B(b,r)$.

[3] This part is from [3, p70]

2. Consider two open balls, $B(a, r)$ and $B(a', r')$ with $r \leq r'$. If they are not disjoint, then there exists $c \in B(a, r) \cap B(a', r')$. Using part 1

$$B(a, r) = B(c, r) \subseteq B(c, r') = B(a', r')$$

giving the required result.

3. The diameter[4] of a subset X of a metric space M is defined to be

$$\sup\{d(x, y) : x, y \in X\}$$

For all $x, y \in B(a, r)$

$$d(x, y) \leq \max\{d(x, a), d(y, a)\} < r$$

Therefore, the supremum of the set $\{d(x, y) : x, y \in B(a, r)\}$ is less than or equal to r. □

To get a further p-adic understanding of what we have discovered, we will see an example[5] of what balls are like using the p-adic derived metric.

Example. We will consider what the open ball $B(0, 1)$ looks like on $(\mathbb{Q}, d_p)$.

$$B(0, 1) = \{x : |x|_p < 1\}$$

Rewriting x as $\frac{a}{b}$, for x to be in the open ball, it has to satisfy

$$\left|\frac{a}{b}\right|_p < 1$$

which is the same as saying

$$v_p(a) > v_p(b)$$

You can rewrite $\frac{a}{b}$ as

$$\frac{p^{v_p(a)}a'}{p^{v_p(b)}b'} = \frac{p^{v_p(a)-v_p(b)}a'}{b'}$$

for some a', b'. This rewriting shows us that x belongs to the ball if, when written as a fraction in lowest terms, the denominator is not divisible by p, but the numerator is.

1.3 The equivalence of absolute values

Definition 1.7. *Two absolute values on a field $\mathbb{F}$ are* equivalent *if they define the same topology.*

This definition is hard to use practically, so the following lemma offers a more accessible route to finding equivalence of absolute values:

Lemma 1.7. *Let $|\cdot|_1$ and $|\cdot|_2$ be two equivalent absolute values on a field $\mathbb{F}$. Then the following statements are equivalent:*

(i) $|\cdot|_1$ *and* $|\cdot|_2$ *are equivalent absolute values.*

(ii) $|x|_1 < 1 \iff |x|_2 < 1 \ \forall \ x \in \mathbb{F}$.

(iii) There exists an $\alpha \in \mathbb{R}_{>0}$ such that for all $x \in \mathbb{F}$, we have $|x|_1 = |x|_2^{\alpha}$.

The proof for this lemma is very technical, and can be found in [1, p42] and [2, pp20-21].

[4] This definition is from [6].

[5] This is based on an exercise from [1, p35]

The following results are about the equivalence of certain absolute values:

Lemma 1.8. *Given two distinct primes, p and q, the p-adic and q-adic absolute values are not equivalent.*

Proof. Note that $|p|_p = \frac{1}{p}$, and $|q|_p = 1$. Using one of the criteria for equivalence from Lemma 1.7, there is no positive real α satisfying

$$|p|_p = \frac{1}{p} = 1^\alpha = |p|_q$$

Therefore, the two absolute values cannot be equivalent. □

Lemma 1.9. *The p-adic absolute value is not equivalent to $|\cdot|_\infty$.*

Proof. This proof is very similar to the one above. Since $|p|_p = \frac{1}{p}$, and $|p|_\infty = p$, there is no positive real α satisfying

$$|p|_p = \frac{1}{p} = p^\alpha = |p|_\infty$$

Therefore, $|\cdot|_p$ and $|\cdot|_\infty$ are not equivalent. □

Theorem 1.10. (Ostrowski's theorem) *Every non-trivial absolute value on $\mathbb{Q}$ is equivalent to $|\cdot|_p$ for some prime p, or for $p = \infty$.*

Note. A trivial absolute value is one of the form

$$|x| = \begin{cases} 0 & \text{if } x = 0 \\ k & \text{if } x \neq 0 \end{cases}$$

for some positive constant k.

In both [1] and [2], they break up the proof of Ostrowski's theorem into two sections:

- showing that all archimedian absolute values are equivalent to $|\cdot|_\infty$.
- showing that all non-archimedian absolute values are equivalent to a p-adic absolute value.

In practical terms, this theorem means that all absolute values that you can use on $\mathbb{Q}$ can be classified into two types.

2 Completing $\mathbb{Q}$

2.1 Creating the completion

Here are some important definitions that are crucial for dealing with the completion of $\mathbb{Q}$:

Definition 2.1. *Let $\mathbb{F}$ be a field, and $|\cdot|$ be an absolute value on $\mathbb{F}$. Then:*

- *A sequence (x_n), with $x_n \in \mathbb{F}$ is called* Cauchy *if, for every ε, there is N such that whenever $n, m > N$, $|x_m - x_n| < \varepsilon$.*

- $\mathbb{F}$ *is* complete *with respect to* $|\cdot|$ *if every Cauchy sequence has a limit in* $\mathbb{F}$.
- *A subset* $S \subset \mathbb{F}$ *is* dense *in* $\mathbb{F}$ *if for every* $x \in \mathbb{F}$ *and for every* $\varepsilon > 0$*, the open ball* $B(x, \varepsilon)$ *has a non-empty intersection with* S.

Definition 2.2. *A* completion[6] *of a metric space* M *is a complete metric space* N *such that* M *is a dense subspace of* N.

The completion of $\mathbb{Q}$ that we are most familiar with is $\mathbb{R}$. It is clear that $\mathbb{R}$ is indeed a complete metric space with respect to $|\cdot|_\infty$, and that $\mathbb{Q}$ is a dense subset of $\mathbb{R}$.

Proposition 2.1. *The field of rational numbers is not complete with respect to any of its non-trivial absolute values.*

It is easy to show that, equipped with $|\cdot|_\infty$, $\mathbb{Q}$ is not complete. Take a sequence of rational numbers that converge to π, such as the sequence obtained by truncating its decimal expansion at the n-th place:

$$3, 3.1, 3.14, 3.141, \ldots$$

Using Ostrowski's theorem, this means that for any archimedian absolute value, the field of rational numbers is not complete. A proof for the non-archimedian case is given in [1, p50].

Because of the fourth condition of the non-archimedian absolute value, we notice that sequences have to satisfy a simplified version of the normal Cauchy property. By checking that adjacent terms get within ε of each other, we can say that the sequence is Cauchy in this case.

Lemma 2.2. *A sequence* $(x_n) \in \mathbb{Q}$ *is Cauchy with respect to a non-archimedian absolute value* $|\cdot|$ *iff*

$$\lim_{n\to\infty} |x_{n+1} - x_n| = 0$$

Note. This property does not hold for Cauchy sequences with archimedian absolute values. Take the sequence $(x_n) = (\sqrt{n})$ with the usual absolute value. Then

$$\lim_{n\to\infty} |x_{n+1} - x_n| = \lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = \lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0$$

but the sequence has no limit.

Proof. Without loss of generality, in this proof, $m = n + t > n$.

- If (x_n) is Cauchy, then $|x_{n+t} - x_n| \to 0$ as $n \to \infty$. Take $t = 1$.
- If $\lim_{n\to\infty} |x_{n+1} - x_n| = 0$, then for every ε, there is N such that whenever $n > N$, $|x_{n+1} - x_n| < \varepsilon$. Consider $|x_m - x_n|$:

$$\begin{aligned} |x_m - x_n| &= |x_{n+t} - x_n| \\ &= |x_{n+t} - x_{n+t-1} + x_{n+t-1} - x_{n+t-2} + \ldots + x_{n+1} - x_n| \\ &\le \max\{|x_{n+t} - x_{n+t-1}|, |x_{n+t-1} - x_{n+t-2}|, \ldots, |x_{n+1} - x_n|\} \end{aligned}$$

[6] This definition is from [5]

For $n > N$

$$\begin{aligned}|x_m - x_n| &\leq \max\{|x_{n+t} - x_{n+t-1}|, |x_{n+t-1} - x_{n+t-2}|, \dots, |x_{n+1} - x_n|\} \\ &< \varepsilon\end{aligned}$$

making (x_n) Cauchy. □

This result is especially useful when looking at p-adic analysis, as it makes convergence easier to deal with.

We are now going to construct the p-adic completion of $\mathbb{Q}$. To do this, we will require some definitions from ring theory:

Definition 2.3. *A* ring[7] $(R, +, \times)$ *is a set R with two binary operations, $+$ (addition) and $\times$ (multiplication), that satisfy the following conditions:*

- $(R, +)$ *is an abelian group.*
- *Multiplication is associative.*
- $a \times (b + c) = (a \times b) + (a \times c)$ *and* $(a + b) \times c = (a \times c) + (b \times c)$ *for all* $a, b, c \in R$.

Definition 2.4. *An* ideal[8] *is an additive subgroup N of a ring R that satisfies*

$$aN \subseteq N$$

$$Nb \subseteq N$$

for all $a, b \in R$.

Definition 2.5. *An ideal I of a ring R is a* maximal ideal[9] *of R if, for some ideal J of R such that $I \subseteq J$, then either $J = I$ or $J = R$.*

With these definitions, we can start to construct the completion. Since we currently know of sequences whose terms get more and more "crowded" but never reach a limit, we will create these limits with these Cauchy sequences.

Definition 2.6. *We define $\mathcal{C}$ to be the set of all Cauchy sequences in $\mathbb{Q}$ with respect to $|\cdot|_p$.*

Proposition 2.3. *Defining operations on $\mathcal{C}$ as*

$$(x_n) + (y_n) = (x_n + y_n)$$

$$(x_n) \cdot (y_n) = (x_n y_n)$$

makes $\mathcal{C}$ a commutative ring with unity.[10]

Proof. Obviously, the multiplicative identity element is the constant sequence $(1, 1, 1, \dots)$. To check that $\mathcal{C}$ is a ring, we need to ensure that the sum and product of two Cauchy sequences is a Cauchy sequence; all other properties of the ring are easy to prove from there.

[7] This definition is from [4, p167]
[8] This is from [4, p241]
[9] This is adapted from [4, p247] and [6]
[10] "with unity" means that the ring contains a multiplicative identity element.

- Fix $\varepsilon > 0$. If (x_n), (y_n) are Cauchy sequences, then there exists N_1, N_2 such that
$$n, m > N_1 \Rightarrow |x_m - x_n| < \frac{\varepsilon}{2}$$
$$n, m > N_2 \Rightarrow |y_m - y_n| < \frac{\varepsilon}{2}$$
Let $N = \max\{N_1, N_2\}$. Then
$$\begin{aligned}|(x_m + y_m) - (x_n + y_n)| &= |(x_m - x_n) + (y_m - y_n)| \\ &\leq |x_m - x_n| + |y_m - y_n| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

- This proof uses the identity
$$(x_m y_m) - (x_n y_n) = x_m(y_m - y_n) + y_n(x_m - x_n)$$
As before, fix $\varepsilon > 0$. If (x_n), (y_n) are Cauchy sequences, then there exists N_1, N_2 such that
$$n, m > N_1 \Rightarrow |x_m - x_n| < \frac{\varepsilon}{2K_1}$$
$$n, m > N_2 \Rightarrow |y_m - y_n| < \frac{\varepsilon}{2K_2}$$
where K_1, K_2 satisfy $|x_m| < K_1$ and $|y_n| < K_2$. We know K_1, K_2 exist since Cauchy sequences are bounded. Let $N = \max\{N_1, N_2\}$. Then
$$\begin{aligned}|(x_m y_m) - (x_n y_n)| &= |x_m(y_m - y_n) + y_n(x_m - x_n)| \\ &\leq |x_m||(y_m - y_n)| + |y_n||(x_m - x_n)| \\ &< \frac{K_1\varepsilon}{2K_1} + \frac{K_2\varepsilon}{2K_2} = \varepsilon\end{aligned}$$
□

$\mathcal{C}$ is not a field; it contains "zero divisors" which are non-zero elements whose product is zero.

Example. Take the two Cauchy sequences $(p, 0, p^2, 0, \ldots)$ and $(0, p, 0, p^2, \ldots)$. Their product is zero.

Note. The ring $\mathcal{C}$ contains representatives of all the elements of $\mathbb{Q}$. The rational number x becomes the sequence $(x, x, x, \ldots)$. This means that there is an inclusion of $\mathbb{Q}$ into $\mathcal{C}$.

We still have the problem that two elements of $\mathcal{C}$ can get arbitrarily close together, and yet seem to be representing two unique elements. We want to be able to make sequences that ought to converge to the same limit equivalent. We achieve this by first looking at null sequences.

Definition 2.7. *We define $\mathcal{N}$ to be the set of all null sequences in $\mathbb{Q}$ with respect to $|\cdot|_p$.*

Proposition 2.4. *The set $\mathcal{N}$ is an ideal of $\mathcal{C}$.*

Proof. Cauchy sequences are bounded; using basic analysis facts, multiplying a null sequence with a bounded sequence results in a null sequence. So, for any $(a_n), (b_n) \in \mathcal{C}$ and for any $(c_n) \in \mathcal{N}$, we get

$$(a_n) \cdot (c_n) \in \mathcal{N}$$

$$(c_n) \cdot (b_n) \in \mathcal{N}$$

making $\mathcal{N}$ an ideal. □

Indeed, $\mathcal{N}$ is a maximal ideal; the proof of this fact is given by Gouvêa, [1, pp52-53].

We now will use the quotient ring structure to make the Cauchy sequences that "converge" to the same limit equivalent as follows:

Definition 2.8. *The* set of p-adic numbers *is defined to be the quotient of the ring* $\mathcal{C}$ *by its maximal ideal* $\mathcal{N}$*, that is*

$$\mathbb{Q}_p = \mathcal{C}/\mathcal{N}$$

Note. The quotient of a ring with an ideal is the analogue to a quotient group formed by a group with a normal subgroup. The elements of $\mathbb{Q}_p = \mathcal{C}/\mathcal{N}$ are equivalence classes of sequences, where two sequences are equivalent if their difference is a null sequence.

In fact, the p-adic numbers, as defined above, form a field, due to the following theorem:

Theorem 2.5. *Let R be a commutative ring with unity. Then R/M is a field iff M is a maximal ideal of R.*

A proof of this is given in Fraleigh's text, [4, p247].

Note. This method of completion is quite commonly used, and it is easy to see that this is one way to derive the real numbers, with Cauchy sequences that use the usual absolute value.

2.2 Confirming the completion

We now need to check that $\mathbb{Q}_p$ is indeed a completion, as defined previously. For this, we require the following statements to be true:

- The p-adic absolute value extends to $\mathbb{Q}_p$, making $(\mathbb{Q}_p, d_p)$ a metric space.
- $\mathbb{Q}$ is a dense subspace of $\mathbb{Q}_p$.
- $\mathbb{Q}_p$ is complete with respect to $|\cdot|_p$.

In this essay, we will prove the first two points; Gouvêa offers a sketch proof of the third fact in [1, p56].

Definition 2.9. *Let $\lambda \in \mathbb{Q}_p$, and (x_n) be a sequence of rational numbers representing λ. We then define:*

$$|\lambda|_p = \lim_{n\to\infty} |x_n|_p$$

To check that this limit exists, consider two cases: $\lambda = 0$ and $\lambda \neq 0$. For $\lambda = 0$, the sequence representing it will be null. Thus, $|x_n|_p \to 0$ as $n \to \infty$, giving $|\lambda|_p = 0$, which fits into our previous definition for $|\cdot|_p$. For non-zero λ, we will require the following lemma:

Lemma 2.6. *Let $(x_n) \in \mathcal{C} \setminus \mathcal{N}$. Then there exists $N \in \mathbb{Z}$ such that, for all $n, m > N$, $|x_n|_p = |x_m|_p$.*

Proof. This is adapted from [1, p54]. (x_n) is not a null sequence, so it is eventually non-zero. That means there is some K_1, c satisfying

$$n > K_1 \Rightarrow |x_n|_p \geq c > 0$$

(x_n) is also Cauchy. This means there is some K_2 so that

$$n, m > K_2 \Rightarrow |x_m - x_n|_p < c$$

Let $K = \max\{K_1, K_2\}$. Then, for all $n, m > K$, $|x_n|_p, |x_m|_p \geq c$, and

$$|x_m - x_n|_p < c \leq \max\{|x_m|_p, |x_n|_p\}$$

Using Lemma 1.5 and the fact that $|x_m - x_n|_p < \max\{|x_m|_p, |x_n|_p\}$, we must have $|x_m|_p = |x_n|_p$. □

This lemma proves the existence of the limit for non-zero λ, since eventually the sequence will have a constant value, so it clearly has a limit.

Lemma 2.7. *The value of $|\lambda|_p$ does not depend on the choice of sequence representing λ.*

Proof. Consider two sequences that represent λ, (x_n) and $(\tilde{x}_n)$. Since they both represent λ, they differ by a null sequence, say (k_n). So

$$x_n = k_n + \tilde{x}_n$$

Applying absolute values and the triangle inequality gives

$$|x_n|_p \leq |k_n|_p + |\tilde{x}_n|_p$$

Taking limits of both sides

$$\lim_{n\to\infty} |x_n|_p \leq \lim_{n\to\infty} |k_n|_p + \lim_{n\to\infty} |\tilde{x}_n|_p = \lim_{n\to\infty} |\tilde{x}_n|_p$$

Similarly, since $\tilde{x}_n = -k_n + x_n$

$$\lim_{n\to\infty} |\tilde{x}_n|_p \leq \lim_{n\to\infty} |-k_n|_p + \lim_{n\to\infty} |x_n|_p = \lim_{n\to\infty} |x_n|_p$$

This results in $\lim_{n\to\infty} |x_n|_p = \lim_{n\to\infty} |\tilde{x}_n|_p$, which proves that different representatives of λ result in the same absolute value. □

Lemma 2.8. *For $\lambda \in \mathbb{Q}_p$, $|\lambda|_p = 0 \iff \lambda = 0$.*

Proof. Let (x_n) be a sequence representing λ. Then

$$|\lambda|_p = 0 \iff \lim_{n\to\infty} |x_n|_p = 0 \iff (x_n) \text{ is a null sequence} \iff \lambda = 0$$ □

Proposition 2.9. $|\cdot|_p : \mathbb{Q}_p \to \mathbb{R}_{\geq 0}$ *is a non-archimedian absolute value.*

Proof. We will check that the function satisfies the four conditions from Definition 1.1. Most of the properties of this absolute value are inherited from the p-adic absolute value when it takes rational values.

1. This follows directly from Lemma 2.8.

2. Let $\lambda, \mu \in \mathbb{Q}_p$, and let $(x_n), (y_n)$ be two sequences representing λ and μ respectively. Then

$$\begin{aligned}|\lambda|_p \cdot |\mu|_p = \lim_{n\to\infty} |x_n|_p \cdot \lim_{n\to\infty} |y_n|_p &= \lim_{n\to\infty} |x_n|_p \cdot |y_n|_p \\ &= \lim_{n\to\infty} |x_n \cdot y_n|_p \\ &= |\lambda \cdot \mu|_p\end{aligned}$$

3. Using $\lambda, \mu, (x_n), (y_n)$ as before

$$\begin{aligned}|\lambda + \mu|_p = \lim_{n\to\infty} |x_n + y_n|_p &\leq \lim_{n\to\infty} |x_n|_p + |y_n|_p \\ &= \lim_{n\to\infty} |x_n|_p + \lim_{n\to\infty} |y_n|_p \\ &= |\lambda|_p + |\mu|_p\end{aligned}$$

4. This is proved very similarly to 3. □

Note. Remember that rational numbers can be represented by constant sequences and the limit of a constant sequence (x) is x. Applying Definition 2.9 confirms that the p-adic absolute value on $\mathbb{Q}_p$ maps elements of $\mathbb{Q}$ to the values we established earlier in this essay.

Now we have established that the p-adic absolute value can be extended to include elements of $\mathbb{Q}_p$, we must check that the image of $\mathbb{Q}$ using the inclusion $x \mapsto (x)$ is dense in $\mathbb{Q}_p$.

Proposition 2.10. *The image of $\mathbb{Q}$ using the inclusion $x \mapsto (x)$ is dense in $\mathbb{Q}_p$.*

Proof. Schikhof offers a very brief proof in his text [2, pp12-13], however, we will be following the more instructive proof offered by Gouvêa [1, pp55-56]. We need to show that, for any open ball in $\mathbb{Q}_p$, there is a constant sequence, i.e. a rational number, in that ball. Fix $\varepsilon > 0$. Let (x_n) be a Cauchy sequence representing λ, and let (y) be the constant sequence $(x_N, x_N, \ldots)$ where N satisfies

$$n, m \geq N \implies |x_m - x_n| < \varepsilon'$$

where $0 < \varepsilon' < \varepsilon$. We want to show that $(y) \in B(\lambda, \varepsilon)$, that is

$$|\lambda - (y)| < \varepsilon$$

Remember that $\lambda - (y)$ can be represented by the sequence $(x_n - x_N)$ and so

$$|\lambda - (y)| = |(x_n - x_N)| = \lim_{n\to\infty} |x_n - x_N|$$

For any $n \geq N$, we have

$$|x_n - x_N| < \varepsilon'$$

Taking the limit gives

$$|\lambda - (y)| = \lim_{n\to\infty} |x_n - x_N| \leq \varepsilon' < \varepsilon \qquad \square$$

Note. Using ε' being less than ε was required in the proof. Despite $|x_n - x_N|$ being less than ε' for all n, there was no way to verify that the limit didn't tend to ε' from below.

Closing remarks

The exploration of p-adic numbers in this essay has taken a more algebraic approach. In many other texts, the p-adic numbers are generated first, and then shown to have the properties we have discovered.

Now we have created the field of p-adic numbers, they can be studied them as one studies the real numbers in basic analysis; one can study convergence, create power series, and look at the functions that can take values in $\mathbb{Q}_p$ and compare them to their real counterparts.

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