The Gamma Function

MA213 Second Year Essay

0504432

April 24, 2007

1 Extending the Factorials

There are many functions which are defined only for natural numbers. One of the simplest is sum of the first n natural numbers:

$$T\colon \mathbb{N} \to \mathbb{N} \qquad T(n) := \sum_{k=1}^{n} k = 1 + 2 + \cdots + (n-1) + n.$$

Simple algebra yields the formula

$$T(n) = \frac{1}{2} n(n+1) \tag{1.1}$$

as rediscovered by Gauss in his childhood [**11**]. This formula does not rely on summation, so we can plug in non-integer values and get, for example,

$$T\left(4\tfrac{1}{2}\right) = \frac{1}{2} \cdot 4\tfrac{1}{2} \cdot 5\tfrac{1}{2} = \frac{99}{8} = 12\tfrac{3}{8}.$$

It makes no sense to talk about the sum of the first $4\frac{1}{2}$ natural numbers, but equation (1.1) affords one possible interpolation between the known values.

In the same way, the factorial function $n!$ is only defined for positive integers:

$$n! := n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$$

together with the convention that $0! = 1$ (it is an empty product). Factorials appear in so many places in mathematics, particularly in analysis and combinatorics, that a formula like (1.1) which would allow computation of $n!$ without having to perform the multiplication was searched for by many mathematicians. However, no "simple" formula like (1.1) exists [**7**].

The question of finding any formula, simple or not, vexed the mathematicians of the eighteenth and nineteenth centuries. It was Leonhard Euler (1707–1783) who first discovered a formula for the factorials. The so-called Gamma function is defined in Rudin [**9**] as follows:

Definition 1.1. *For* $0 < x < \infty$,

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} \,\mathrm{d}t. \tag{1.2}$$

But what does definition 1.1 mean? We are used to integrating over a closed interval $[a, b]$, not over a range like $[0, \infty)$. What is meant by this notation is an *improper integral*:

$$\int_a^\infty f(t) \,\mathrm{d}t := \lim_{k \to \infty} \int_a^k f(t) \,\mathrm{d}t.$$

When $0 < x < 1$, there is an additional problem lurking in (1.2): t^{x-1} is undefined at $x = 0$. So in fact we must also take another limit:

$$\int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t := \lim_{k\to\infty}\lim_{h\to 0}\int_h^k t^{x-1}e^{-t}\,\mathrm{d}t.$$

Fortunately, many of the familiar properties of integrals hold for improper integrals. But we must first verify that our definition makes sense, i.e. that the integral in definition 1.1 does actually converge. To do so, we make use of a comparison test for improper integrals:

Lemma 1.2. *If $0 \le f(x) \le g(x)$ for all $x \in (a,b)$, where $a, b \in [-\infty, \infty]$, and the improper integral $\int_a^b g(x)\,\mathrm{d}x$ converges, then the improper integral $\int_a^b f(x)\,\mathrm{d}x$ converges also.*

This is intuitively obvious: if the area under the graph of a function g is finite, then a subset of that area (the area under the graph of f) will also be finite. (This is much like the comparison test for infinite series: if $0 \le a_n \le b_n$ for all $n \in \mathbb{N}$, and $\sum b_n$ converges, then $\sum a_n$ converges too.) A formal proof is given in Walker [**12**, thm. 5.3].

Lemma 1.3. $\displaystyle\int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t$ *converges for all $x \in (0,\infty)$.*

Proof. Firstly, we split the integral into

$$\int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t = \int_0^1 t^{x-1}e^{-t}\,\mathrm{d}t + \int_1^\infty t^{x-1}e^{-t}\,\mathrm{d}t$$

and show that each of the two integrals on the right-hand side converge individually. In each case we find $f(t)$ such that $0 \le t^{x-1}e^{-t} \le f(t)$, and then show that the integral of $f(t)$ over the domain is finite, and hence so is the integral of $t^{x-1}e^{-t}$.

For the first integral, since $e^{-t} \le 1$ for $t \ge 0$, we have that $0 \le t^{x-1}e^{-t} \le t^{x-1}$ for $t > 0$. Then

$$0 \le \int_0^1 t^{x-1}e^{-t}\,\mathrm{d}t \le \int_0^1 t^{x-1}\,\mathrm{d}t = \lim_{a\to 0^+}\int_a^1 t^{x-1}\,\mathrm{d}t = \lim_{a\to 0^+}\left[\frac{t^x}{x}\right]_{t=a}^1 = \lim_{a\to 0^+}\left(\frac{1}{x} - \frac{a^x}{x}\right).$$

Now, if $x > 0$, then $a^x \to 0$ as $a \to 0^+$, so the integral $\int_0^1 t^{x-1}\,\mathrm{d}t$ converges to $\frac{1}{x}$. Hence $\int_0^1 t^{x-1}e^{-t}\,\mathrm{d}t$ converges.

For the second integral, note that $\lim_{t\to\infty} t^r e^{-t/2} = 0$ for any $r \in \mathbb{R}$. Hence for any $x > 0$, there exists k_x such that $0 \le t^{x-1}e^{-t/2} \le 1$ for $t \ge k_x$. Now fix $x > 0$, and split the integral at k_x:

$$\int_1^\infty t^{x-1}e^{-t}\,\mathrm{d}t = \int_1^{k_x} t^{x-1}e^{-t}\,\mathrm{d}t + \int_{k_x}^\infty t^{x-1}e^{-t}\,\mathrm{d}t.$$

The first of these terms is simply a finite integral. For the second, when $t \ge k_x$ we have that $t^{x-1}e^{-t} = (t^{x-1}e^{-t/2})e^{-t/2} \le e^{-t/2}$, and so

$$\int_{k_x}^\infty t^{x-1}e^{-t}\,\mathrm{d}t \le \int_{k_x}^\infty e^{-t/2}\,\mathrm{d}t = \lim_{a\to\infty}\left[-2e^{-t/2}\right]_{t=k_x}^a = \lim_{a\to\infty}\left(2e^{-k_x/2} - 2e^{-a/2}\right) = 2e^{-k_x/2}.$$

This shows that the second term is convergent, and so $\int_1^\infty t^{x-1}e^{-t}\,\mathrm{d}t$ converges. This means that $\int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t$ converges for all x with $0 < x < \infty$, by lemma 1.2. □

Having shown that $\Gamma(x)$ is well-defined, we must consider what the definition actually means. It is all very well to say that Γ extends the factorial function to $0 < x < \infty$, but this is not at all obvious from the definition. In order to show this, we first need a preliminary result:

Proposition 1.4. *For all $x > 0$, $\Gamma(x+1) = x\Gamma(x)$.*

Proof. From (1.2), $\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,\mathrm{d}t$. Integrating by parts gives

$$\begin{aligned}\Gamma(x+1) &= \left[-t^x e^{-t}\right]_0^\infty - \int_0^\infty x t^{x-1}(-e^{-t})\,\mathrm{d}t \\ &= \lim_{k\to\infty}\left[-e^{-t}t^x\right]_0^k + \int_0^\infty x t^{x-1}e^{-t}\,\mathrm{d}t \\ &= \lim_{k\to\infty}\left(-e^{-k}k^x\right) + x\int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t\end{aligned}$$

Now as $k \to \infty$, $k^x e^{-k} \to 0$, so we get

$$\Gamma(x+1) = x\int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t = x\Gamma(x). \qquad \square$$

We are now ready to show:

Theorem 1.5. *For all $n \in \mathbb{N}$, $\Gamma(n+1) = n!$.*

Proof. By proposition 1.4, we have

$$\begin{aligned}\Gamma(n+1) &= n\cdot\Gamma(n) \\ &= n\cdot(n-1)\cdot\Gamma(n-1) \\ &\;\;\vdots \\ &= n\cdot(n-1)\cdot\dots\cdot 2\cdot 1\cdot\Gamma(1) \\ &= n!\cdot\Gamma(1).\end{aligned}$$

Hence it suffices to show that $\Gamma(1) = 1 = 0!$. By definition 1.1, we have

$$\Gamma(1) = \int_0^\infty e^{-t}\,\mathrm{d}t = \lim_{k\to\infty}\left[-e^{-t}\right]_0^k = \lim_{k\to\infty}\left(-e^{-k}+1\right) = 1. \qquad \square$$

The modern notation is due to Adrien–Marie Legendre (1752–1833). It is unfortunate that $\Gamma(n+1) = n!$ rather than $\Gamma(n) = n!$; the reasons have become lost in the mists of time.

2 Convexity and the Gamma Function

Theorem 1.5 shows that Γ gives us a way of extending the factorial function from the natural numbers to all real $x > -1$. But it is not the only way. If we simply draw any curve through the points $(1,1)$, $(2,2)$, $(3,6)$, $(4,24)$, ..., then it will return factorials at integer values. Euler's answer was to create the Gamma function as in definition 1.1, like some *deus ex machina*. Jacques Hadamard (1865–1963) proposed the following alternative [**7**, (34)]:

$$F(x) := \frac{1}{\Gamma(1-x)}\frac{\mathrm{d}}{\mathrm{d}x}\log\left(\frac{\Gamma\left(\frac{1-x}{2}\right)}{\Gamma\left(1-\frac{x}{2}\right)}\right).$$

This does indeed give $F(n+1) = n!$ for every natural number n. So why do we consider Euler's solution as *the* solution?

We want a function which extends the factorial function to all real $x > -1$, which the Gamma function does. But this requirement does not, of itself, restrict us enough; that is, there are other functions, besides the Gamma function, which satisfy $f(n+1) = n!$ for all natural numbers n. So, we seek a condition which means that any such function cannot be anything else but the Gamma function.

First of all, we note that proposition 1.4 does not only apply for natural numbers x; the formula $\Gamma(x+1) = x\Gamma(x)$ is valid for any $x > 0$. This is therefore stronger than only requiring $f(n+1) = n!$, since it means that the values of Γ in any range $[x, x+1]$ determine Γ on the whole real line. This restricts us somewhat, but not enough; Davis [**7**, (35)] provides the example

$$G(x) := \begin{cases} \frac{1}{x} & 0 < x \leq 1 \\ 1 & 1 \leq x \leq 2 \\ x-1 & 2 \leq x \leq 3 \\ (x-1)(x-2) & 3 \leq x \leq 4 \\ \vdots & \end{cases}$$

which satisfies the relation in proposition 1.4 and has $G(1) = 1$, but clearly is not the Gamma function.

To completely characterise the Gamma function among the possible extensions of the factorial function, we introduce the notion of *convexity*. This is a very intuitive concept: a subset $X \subset \mathbb{R}^n$ is convex if, when you pick any two points $A, B \in X$, every point lying on the line joining A to B is also in X.

For example, consider the graph of $f(x) = x^2$. The set of points lying on or above the graph of f is a subset of $\mathbb{R}^2$: moreover, this set is convex. In general, we call a function convex if the set of points lying on or above the graph of the function (known as the *epigraph*) is a convex set. We define this formally as follows:

Definition 2.1. *A function $f\colon (a,b) \to \mathbb{R}$ is called* convex *if for any $x, y \in (a,b)$, we have*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

for all $\lambda \in (0,1)$.

Here we have parametrised the interval (x, y) as $\{\lambda x + (1-\lambda)y \mid 0 < \lambda < 1\}$; what this definition means is that, as you move from x to y, the line joining $(x, f(x))$ to $(y, f(y))$ always lies above the graph of f. Convexity is a smoothness condition on a function; any convex function on an open interval must be continuous, as we show now:

Lemma 2.2. *Any convex function $f\colon (a,b) \to \mathbb{R}$ is continuous.*

Proof. This proof is taken from Rudin [**10**, thm. 3.2]. The idea of the proof is best conveyed in geometric language, and, as Rudin slyly notes, "those who may worry that this is not 'rigorous' are invited to transcribe it in terms of epsilons and deltas".

Suppose $a < s < x < y < t < b$. Write S for the point $(s, f(s))$, and similarly for x, y and t.

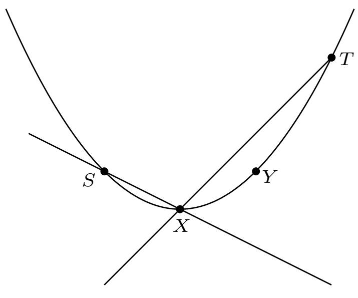

Figure 1: A convex function.

As can be seen from figure 1, convexity implies that X must lie on or below the line SY, hence Y is on or above the line through S and X; also, Y is on or below the line XT. As $y \to x^+$, the point Y is sandwiched between these two lines, and hence $f(y) \to f(x)$. Left-hand limits are handled similarly, and continuity of f follows. □

Note that the openness of the domain is necessary: for instance $f(x) = 0$ for $x \in [0, 1)$ and $f(1) = 1$ is a convex function on $[0, 1]$ which is not continuous [**10**]. Clearly, however, such discontinuities can only occur at the endpoints of the interval.

Given two points x and y in the domain of a real function f, we form the *difference quotient* $\frac{f(y)-f(x)}{y-x}$, representing the slope of the chord from $(x, f(x))$ to $(y, f(y))$. With convex functions, the difference quotient always increases as we increase x and y:

Proposition 2.3. *If $f\colon (a, b) \to \mathbb{R}$ is convex and if $a < s < t < u < b$, then*

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}$$

Proof. Let $a < s < t < u < b$. Then

$$f(\lambda s + (1 - \lambda)u) \leq \lambda f(s) + (1 - \lambda)f(u)$$

Let $\lambda = \frac{u-t}{u-s}$. Then $\lambda s + (1 - \lambda)u = u + \lambda(s - u) = t$, and so

$$f(t) \leq f(u) + \frac{u - t}{u - s} \cdot (f(s) - f(u)). \tag{2.1}$$

Rearranging (2.1) gives $\frac{f(u)-f(s)}{u-s} \leq \frac{f(u)-f(t)}{u-t}$. Furthermore, writing $\frac{u-t}{u-s} = 1 + \frac{s-t}{u-s}$ in (2.1) gives

$$f(t) \leq f(s) + \frac{s - t}{u - s}(f(s) - f(u))$$

which rearranged gives $\frac{f(t)-f(s)}{t-s} \leq \frac{f(u)-f(s)}{u-s}$. Hence

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}$$

as required. □

Consider again the example of $f(x) = x^2$. Now, if we take logarithms of both sides, we get $\log f = 2 \log x$. Now, $\log x$ is not a convex function: for each pair of points, the line joining them lies *below* the graph, not above it; so $\log x$ is in fact *concave*, not convex. So $\log f$ is not convex either. But consider $g(x) = e^x$, which is convex; in this case $\log g = x$, which is also convex. We say that g is *log-convex*; this is a stronger property than convexity, as we show now:

Proposition 2.4. *Given a function $f\colon (a, b) \to \mathbb{R}$, if $\log f$ is convex, then so is f itself.*

To prove this we need a preliminary lemma:

Lemma 2.5. *Any increasing convex function of a convex function is convex.*

Proof. Let $f\colon (a, b) \to (h, k)$ be convex and let $g\colon (h, k) \to \mathbb{R}$ be convex and increasing, i.e. $x \leq y \implies g(x) \leq g(y)$. By the convexity of f, for any $a < x < y < b$ we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for any $\lambda \in [0, 1]$. Now as g is increasing, applying g to both sides preserves the inequality. Hence by the convexity of g, we get

$$g(f(\lambda x + (1 - \lambda)y)) \leq g(\lambda f(x) + (1 - \lambda)f(y)) \leq \lambda g(f(x)) + (1 - \lambda)g(f(y)).$$

Hence $g \circ f$ is convex. □

Proof of Proposition 2.4. If $\log f$ is convex, then by lemma 2.5, since e^x is increasing and convex, we have that $e^{\log f} = f$ is also convex. □

We can now return to the Gamma function. Figure 2 shows the graph of Γ:

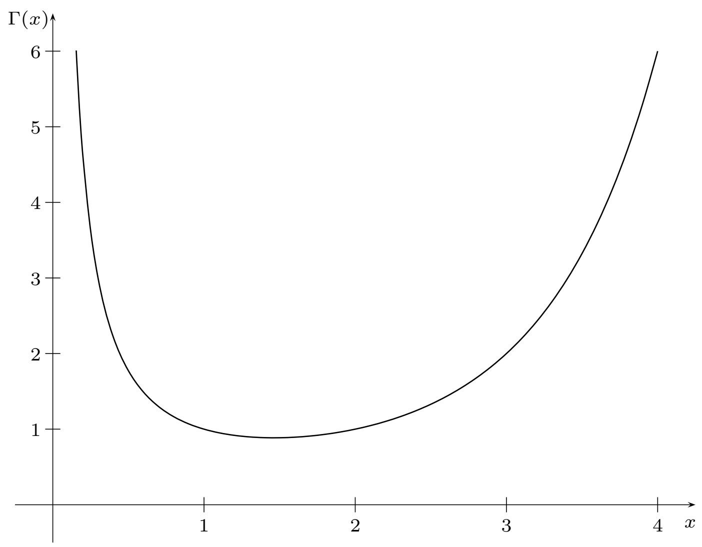

Figure 2: The graph of $\Gamma(x)$.

It is plain to see from figure 2 that Γ is convex. However, Γ in fact increases so steeply as $x \to \infty$ that:

Theorem 2.6. $\Gamma\colon (0, \infty) \to \mathbb{R}$ *is log-convex.*

The graph of $\log \Gamma$ is shown in figure 3.

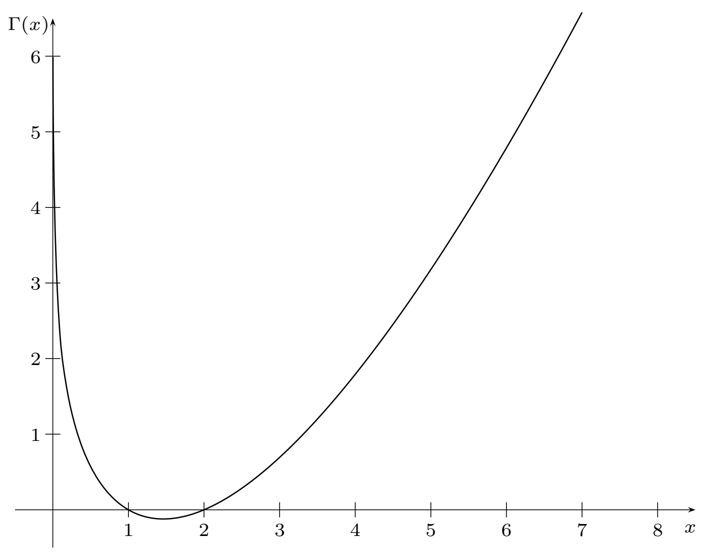

Figure 3: The graph of $\log \Gamma(x)$.

To prove that $\log\Gamma$ is convex (which will imply that Γ is convex) we need an inequality known as *Hölder's inequality*:

Lemma 2.7 (Hölder's inequality)**.** *Let p and q be positive real numbers such that $\frac{1}{p}+\frac{1}{q}=1$. Then for any integrable functions $f, g\colon [a,b]\to\mathbb{R}$, we have*

$$\left|\int_a^b f(x)g(x)\,\mathrm{d}x\right| \le \left(\int_a^b |f|^p\,\mathrm{d}x\right)^{1/p}\left(\int_a^b |g|^q\,\mathrm{d}x\right)^{1/q}.$$

Proofs of Hölder's inequality can be found in Rudin [**10**, thm. 3.5] and Capiński and Kopp [**6**, thm. 5.21].

Proof of Theorem 2.6. Let $1<p<\infty$ and $\frac{1}{p}+\frac{1}{q}=1$. As in Rudin [**9**, thm. 8.18(c)], consider

$$\begin{aligned}
\Gamma\left(\frac{x}{p}+\frac{y}{q}\right) &= \int_0^\infty t^{(x/p+y/q-1)}\,e^{-t}\,\mathrm{d}t \\
&= \int_0^\infty t^{x/p}\,t^{y/q}\,t^{-1/p}\,t^{-1/q}\,e^{-t/p}\,e^{-t/q}\,\mathrm{d}t && \text{as } \tfrac{1}{p}+\tfrac{1}{q}=1 \\
&= \int_0^\infty \left(t^{x-1}e^{-t}\right)^{1/p}\left(t^{y-1}e^{-t}\right)^{1/q}\,\mathrm{d}t \\
&\le \left(\int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t\right)^{1/p}\left(\int_0^\infty t^{y-1}e^{-t}\,\mathrm{d}t\right)^{1/q} && \text{by lemma 2.7} \\
&= \Gamma(x)^{1/p}\,\Gamma(y)^{1/q}.
\end{aligned}$$

Let $\lambda=\frac{1}{p}$, and hence $1-\lambda=\frac{1}{q}$. Then $\lambda\in(0,1)$, and

$$\begin{aligned}
&\Gamma(\lambda x+(1-\lambda)y) \le \Gamma(x)^\lambda\Gamma(y)^{1-\lambda} \\
\implies &\log\Gamma(\lambda x+(1-\lambda)y) \le \log\left[\Gamma(x)^\lambda\Gamma(y)^{1-\lambda}\right] \\
&\qquad = \lambda\log\Gamma(x)+(1-\lambda)\log\Gamma(y).
\end{aligned}$$

for any $x,y\in(0,\infty)$. Hence $\log\Gamma$ is convex. □

At this point one may wonder why the log-convexity of Γ is of any use. It is thus a fascinating theorem of Bohr and Mullerup that proposition 1.4, theorem 1.5 and theorem 2.6 characterise Γ completely:

Theorem 2.8 (Bohr–Mullerup)**.** *If $f\colon(0,\infty)\to(0,\infty)$ satisfies*
1. *$f(1)=1$,*
2. *$f(x+1)=xf(x)$, and*
3. *$\log f$ is convex,*

then $f(x)=\Gamma(x)$ for all $x\in(0,\infty)$.

Essentially, having fixed the integer values of $f(x)$, the convexity of $\log f(x)$ restricts the growth of f in such a way as it must be the Gamma function.

Proof. We follow the elegant proof in Rudin [**9**, thm. 8.19]. Since we have already shown that Γ satisfies conditions 1 to 3, it suffices to prove that $f(x)$ is uniquely determined by these conditions. Furthermore, by condition 2 it is enough to prove this only for $x\in(0,1)$.

Set $\varphi=\log f$. Condition 1 implies that $\varphi(1)=0$. Taking logarithms, condition 2 becomes

$$\varphi(x+1)=\varphi(x)+\log x. \tag{2.2}$$

Condition 3 means that φ is convex.

Let $0 < x < 1$, and let $n \in \mathbb{N}$. Consider the difference quotients of φ as in proposition 2.3; let $s = n$, $t = n+1$, and $u = n+1+x$ to get

$$\varphi(n+1) - \varphi(n) \leq \frac{\varphi(n+1+x) - \varphi(n+1)}{x}.$$

Using the difference quotients for $s = n+1$, $t = n+1+x$, $u = n+2$ gives

$$\frac{\varphi(n+1+x) - \varphi(n+1)}{x} \leq \varphi(n+2) - \varphi(n+1).$$

Note that $\varphi(n+1) - \varphi(n) = \log n$, and $\varphi(n+2) - \varphi(n+1) = \log(n+1)$. So combining these gives

$$\log n \leq \frac{\varphi(n+1+x) - \varphi(n+1)}{x} \leq \log(n+1).$$

Now, repeatedly applying (2.2) gives

$$\begin{aligned}
\varphi(x+n+1) &= \varphi(x+n) + \log(x+n) \\
&= \varphi(x+n-1) + \log(x+n) + \log(x+n-1) \\
&= \varphi(x+n-1) + \log(x+n)(x+n-1) \\
&= \varphi(x+n-2) + \log(x+n)(x+n-1)(x+n-2) \\
&\ \ \vdots \\
&= \varphi(x) + \log\left[(x+n)(x+n-1)\dots(x+1)x\right].
\end{aligned}$$

Also by (2.2), we have $\varphi(n+1) = \log(n!)$. So

$$\frac{\varphi(n+1+x) - \varphi(n+1)}{x} = \frac{1}{x}\left[\varphi(x) + \log\left[(x+n)\dots(x+1)x\right] - \log(n!)\right].$$

This gives

$$\log n \leq \frac{1}{x}\left[\varphi(x) + \log\left[(x+n)\dots(x+1)x\right] - \log(n!)\right] \leq \log(n+1).$$

Multiplying through by x gives

$$\log n^x \leq \varphi(x) + \log\left[(x+n)\dots(x+1)x\right] - \log(n!) \leq \log(n+1)^x.$$

Subtracting $\log n^x$ from each expression gives

$$0 \leq \varphi(x) + \log\left[(x+n)\dots(x+1)x\right] - \log(n!) - \log n^x \leq \log(n+1)^x - \log n^x.$$

Simplifying each of these expressions gives

$$0 \leq \varphi(x) - \log\left[\frac{n!n^x}{x(x+1)\dots(x+n)}\right] \leq x\log\left(1 + \frac{1}{n}\right).$$

Having obtained this, we let $n \to \infty$, where we see that $\log\left(1 + \frac{1}{n}\right) \to \log 1 = 0$, and hence

$$\varphi(x) = \lim_{n\to\infty} \log\left[\frac{n!n^x}{x(x+1)\dots(x+n)}\right]$$

In any case φ is uniquely determined, and the proof is complete. □

From the last equation in this proof, we obtain the following corollary:

Corollary 2.9. *For all* $0 < x < \infty$, $\Gamma(x) = \lim_{n\to\infty} \left[\frac{n!n^x}{x(x+1)\dots(x+n)}\right]$.

This gives us an alternative form of $\Gamma(x)$ which will come in useful later.

Proof. The proof of theorem 2.8 shows that

$$\varphi(x) = \lim_{n\to\infty} \log \left[\frac{n!n^x}{x(x+1)\dots(x+n)}\right]$$

for $0 < x < 1$. As log is continuous, we can exchange the log and the lim to get

$$\varphi(x) = \log \lim_{n\to\infty} \left[\frac{n!n^x}{x(x+1)\dots(x+n)}\right].$$

Exponentiating both sides gives

$$\Gamma(x) = \lim_{n\to\infty} \left[\frac{n!n^x}{x(x+1)\dots(x+n)}\right] \tag{2.3}$$

for $0 < x < 1$. (2.3) also holds for $x = 1$, since then we have

$$\Gamma(1) = \lim_{n\to\infty} \left[\frac{n!n}{1\cdot 2\cdot\dots\cdot(n+1)}\right] = 1.$$

So (2.3) holds for $0 < x \le 1$. Using proposition 1.4, we see that

$$\begin{aligned}\Gamma(x+1) &= x \lim_{n\to\infty} \left[\frac{n!n^x}{x(x+1)\dots(x+n)}\right] \\ &= \lim_{n\to\infty} \frac{x+n+1}{n} \left[\frac{n!n^{x+1}}{(x+1)\dots(x+n)(x+n+1)}\right] \\ &= \lim_{n\to\infty} \left[1+\frac{1+x}{n}\right] \cdot \lim_{n\to\infty} \left[\frac{n!n^{x+1}}{(x+1)\dots(x+n)(x+n+1)}\right] \\ &= \lim_{n\to\infty} \left[\frac{n!n^{x+1}}{(x+1)\dots(x+n)(x+n+1)}\right].\end{aligned}$$

From this we get that (2.3) holds for $1 < x \le 2$ as well; repeatedly applying this argument gives that (2.3) applies for all $x > 0$, as required. ☐

3 The Beta Function

Our quest would seem to be at an end: the Gamma function is the only extension of the factorial function which is log-convex. It should be emphasised that it took mathematicians a long time to figure this out; as Davis [**7**] succinctly puts it: "The proof: one page. The discovery: 193 years."

However, log-convexity seems somewhat contrived as a reason for favouring the Gamma function among possible extensions of the factorial function. We now consider various integrals associated with the Gamma function and show its use in evaluating a number of integrals, as well as some infinite products, and we will see that the real reason for favouring the Gamma function is its ubiquity.

The integral in the definition of the Gamma function (definition 1.1) is known as *Euler's second integral* [**4**]. *Euler's first integral* is another integral related to the Gamma function, which he also proposed:

Definition 3.1. *For $x, y > 0$, define*

$$B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t.$$

This is also known as the Beta function [**9**]. In order that this definition makes sense, we must verify that the integral actually exists, since if $x < 1$ then the integrand blows up at 0, and if $y < 1$ then the integrand blows up at 1. This is again an improper integral, and we prove that it converges now:

Lemma 3.2. $\displaystyle\int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t$ *converges for all $x, y \in (0, \infty)$.*

Proof. As in the proof of lemma 1.3, we split the integral as

$$\int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t = \int_0^{1/2} t^{x-1}(1-t)^{y-1}\,\mathrm{d}t + \int_{1/2}^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t$$

and use lemma 1.2 to prove convergence of each separately.

For the first integral, note that $t^{x-1}(1-t)^{y-1} \leq t^{x-1}$ for $0 < t \leq \frac{1}{2}$, so

$$\int_0^{1/2} t^{x-1}(1-t)^{y-1}\,\mathrm{d}t \leq \int_0^{1/2} t^{x-1}\,\mathrm{d}t = \lim_{a\to 0^+}\left[\frac{t^x}{x}\right]_{t=a}^{1/2} = \lim_{a\to 0^+}\left(\frac{\left(\frac{1}{2}\right)^x - a^x}{x}\right)$$

Now, if $x > 0$, then $a^x \to 0$ as $a \to 0^+$, so the first integral is finite if $x > 0$.

Similarly, for the second integral, note that $t^{x-1}(1-t)^{y-1} \leq (1-t)^{y-1}$ for $\frac{1}{2} \leq t < 1$, so

$$\int_{1/2}^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t \leq \int_{1/2}^1 (1-t)^{y-1}\,\mathrm{d}t = \lim_{a\to 1^-}\left[-\frac{(1-t)^y}{y}\right]_{t=1/2}^{a} = \lim_{a\to 1^-}\left(\frac{\left(\frac{1}{2}\right)^y - (1-a)^y}{x}\right)$$

Now, if $y > 0$, then $(1-a)^y \to 0$ as $a \to 1^-$, so the second integral is finite if $y > 0$. Thus $\int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t$ converges for all $x, y > 0$. ☐

At first sight, the Beta function does not appear to be related to the Gamma function at all; for one, the Beta function is a function of two variables, not of one variable. In light of this, the following theorem is thus somewhat surprising:

Theorem 3.3. *For all $x, y > 0$, we have*

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{3.1}$$

This is in fact a consequence of the Bohr–Mullerup theorem (theorem 2.8): we will show that $B(x,y)$ is log-convex, and hence so is $\frac{B(x,y)\Gamma(x+y)}{\Gamma(y)}$; and since this last expression satisfies the multiplication formula and takes the value 1 when $x = 1$, it must be equal to $\Gamma(x)$.

Proof. The following argument is based on the proofs in Rudin [**9**, thm. 8.20] and Artin [**4**, pp. 18–19].

Consider $f(x) = \frac{\Gamma(x+y)}{\Gamma(y)}B(x,y)$ as a function of x for each fixed y, where $x, y \in (0, \infty)$. We show that $f(x) = \Gamma(x)$ using the Bohr–Mullerup theorem (theorem 2.8). Firstly,

$$f(1) = \frac{\Gamma(1+y)}{\Gamma(y)}B(1,y) = yB(1,y)$$

by proposition 1.4. However,

$$B(1,y) = \int_0^1 (1-t)^{y-1}\,\mathrm{d}t = \int_0^1 u^{y-1}\,\mathrm{d}u = \left.\frac{u^y}{y}\right|_0^1 = \frac{1}{y}$$

using the substitution $u = 1-t$. Hence $f(1) = 1$. Furthermore, consider

$$B(x+1,y) = \int_0^1 t^x (1-t)^{y-1}\,\mathrm{d}t = \int_0^1 \left(\frac{t}{1-t}\right)^x (1-t)^{x+y-1}\,\mathrm{d}t.$$

Integrating the second integral by parts (differentiating the first term and integrating the second) yields

$$\begin{aligned} B(x+1,y) &= \left[-\frac{t^x(1-t)^y}{x+y}\right]_{t=0}^1 - \int_0^1 \frac{-(1-t)^{x+y}}{x+y} \cdot \frac{xt^{x-1}}{(1-t)^{x+1}}\,\mathrm{d}t \\ &= \frac{x}{x+y}\int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t = \frac{x}{x+y}B(x,y). \end{aligned}$$

Hence

$$\begin{aligned} f(x+1) &= \frac{\Gamma(x+y+1)}{\Gamma(y)}B(x+1,y) \\ &= \frac{(x+y)\Gamma(x+y)}{\Gamma(y)} \cdot \frac{x}{x+y}B(x,y) = x \cdot \frac{\Gamma(x+y)}{\Gamma(y)}B(x,y) = xf(x). \end{aligned}$$

It remains to show that $\log f$ is convex. The proof follows the same lines as that of theorem 2.6. Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Consider

$$f\left(\frac{u}{p} + \frac{v}{q}\right) = \frac{\Gamma\left(\frac{u}{p} + \frac{v}{q} + y\right)}{\Gamma(y)} B\left(\tfrac{u}{p} + \tfrac{v}{q}, y\right).$$

Now

$$\begin{aligned} \Gamma\left(\tfrac{u}{p} + \tfrac{v}{q} + y\right) &= \int_0^\infty t^{(u/p\,+\,v/q\,+y-1)}\,e^{-t}\,\mathrm{d}t \\ &= \int_0^\infty \left(t^{u+y-1}e^{-t}\right)^{1/p} \left(t^{v+y-1}e^{-t}\right)^{1/q}\,\mathrm{d}t && \text{as } \tfrac{1}{p} + \tfrac{1}{q} = 1 \\ &\le \left(\int_0^\infty t^{u+y-1}e^{-t}\,\mathrm{d}t\right)^{1/p} \left(\int_0^\infty t^{v+y-1}e^{-t}\,\mathrm{d}t\right)^{1/q} && \text{by lemma 2.7} \\ &= \Gamma(u+y)^{1/p}\,\Gamma(v+y)^{1/q} \end{aligned}$$

and

$$\begin{aligned} B\left(\tfrac{u}{p} + \tfrac{v}{q}, y\right) &= \int_0^1 t^{(u/p\,+\,v/q\,-\,1)}(1-t)^{y-1}\,\mathrm{d}t \\ &= \int_0^1 \left(t^u(1-t)^{y-1}\right)^{1/p} \left(t^v(1-t)^{y-1}\right)^{1/q} && \text{as } \tfrac{1}{p} + \tfrac{1}{q} = 1 \\ &\le \left(\int_0^1 t^u(1-t)^{y-1}\,\mathrm{d}t\right)^{1/p} \left(\int_0^1 t^v(1-t)^{y-1}\,\mathrm{d}t\right)^{1/q} && \text{by lemma 2.7} \\ &= B(u,y)^{1/p}B(v,y)^{1/q}. \end{aligned}$$

So

$$f\left(\frac{u}{p}+\frac{v}{q}\right) \le \frac{1}{\Gamma(y)}\left[\Gamma(u+y)B(u,y)\right]^{1/p}\left[\Gamma(v+y)B(v,y)\right]^{1/q}$$
$$= \left[\frac{\Gamma(u+y)}{\Gamma(y)}B(u,y)\right]^{1/p}\left[\frac{\Gamma(v+y)}{\Gamma(y)}B(v,y)\right]^{1/q} = [f(u)]^{1/p}[f(v)]^{1/q}.$$

Let $\lambda = \frac{1}{p}$, and hence $1-\lambda = \frac{1}{q}$. Then $\lambda \in (0,1)$, and

$$\begin{aligned} & f(\lambda u + (1-\lambda)v) \le f(u)^{\lambda} f(v)^{1-\lambda} \\ \implies & \log f(\lambda u + (1-\lambda)v) \le \log\left[f(u)^{\lambda} f(v)^{1-\lambda}\right] \\ & \qquad = \lambda \log f(u) + (1-\lambda)\log f(v). \end{aligned}$$

for any $u, v \in (0,\infty)$. Hence $\log f$ is convex, and the proof is complete. □

By substituting $x = y = \frac{1}{2}$ into (3.1), we obtain the following wonderful corollary:

Corollary 3.4. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Proof. By substituting $t = \sin^2\theta$, (3.1) turns into

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = 2\int_0^{\pi/2} (\sin\theta)^{2x-1}(\cos\theta)^{2y-1}\,\mathrm{d}\theta.$$

The special case $x = y = \frac{1}{2}$ gives

$$\left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = 2\int_0^{\pi/2} \mathrm{d}\theta = \pi.$$

Since Γ is positive for all $x \in (0,\infty)$, we have $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ as required. □

Furthermore, using this, we can recover the value of an integral fundamental to all of probability theory:

Proposition 3.5. $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$.

Proof. We have $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. By definition,

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^{\infty} t^{-1/2}e^{-t}\,\mathrm{d}t.$$

Substituting $t = x^2$, $\mathrm{d}t = 2x\,\mathrm{d}x$ gives

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^{\infty} x^{-1}e^{-x^2}2x\,\mathrm{d}x = \sqrt{\pi}.$$

Hence

$$\int_0^{\infty} e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}.$$

As e^{-x^2} is symmetric about $x = 0$, we have

$$\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = 2\int_0^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}.$$ □

4 Euler's Reflection Formula

We have seen that the Gamma and Beta functions are closely related. However, by manipulating the form of $B(x, y)$, in certain cases we can evaluate $B(x, y)$ explicitly. Corollary 3.4 is, in fact, merely a special case of Euler's *reflection formula*:

Theorem 4.1 (Euler's reflection formula)**.** *For* $0 < x < 1$, $\Gamma(x)\Gamma(1-x) = \dfrac{\pi}{\sin(\pi x)}$.

(We require $0 < x < 1$ so that $y > 0$.)

There are many different proofs of this statement, none of them particularly simple, and many of which resort to complex integration. The following (long) real-analytic proof is originally due to Richard Dedekind (1831–1916), in one of his earliest papers ([**8**], in German) of 1853. His proof is presented in exercise 16 of Andrews et al. [**3**, ch. 1], and it is that which we follow here.

Proof. Firstly, set $\phi(x) = \Gamma(x)\Gamma(1-x)$. Noting that $\Gamma(1) = 1$, we substitute $y = 1 - x$ in equation (3.1) to give

$$\phi(x) = \int_0^1 u^{x-1}(1-u)^{-x}\,\mathrm{d}u.$$

Substitute $u = \frac{t}{t+1}$, or in other words $t = \frac{u}{1-u}$, to obtain

$$\phi(x) = \int_0^\infty \frac{t^{x-1}}{1+t}\,\mathrm{d}t.$$

Our strategy is to show that ϕ satisfies a differential equation, and then to solve this equation to find the required expression for $\phi(x)$, that is $\phi(x) = \frac{\pi}{\sin(\pi x)}$.

The method by which we obtain the differential equation for ϕ is somewhat roundabout, and involves the manipulation of many integrals. We will not worry about issues of convergence here; all the improper integrals that we consider here do converge, and in such a way that we can change the order of integration:

$$\int_0^\infty \int_0^\infty f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^\infty \int_0^\infty f(x, y)\,\mathrm{d}y\,\mathrm{d}x.$$

Consider the integral $\displaystyle\int_0^\infty \frac{t^{x-1}}{st+1}\,\mathrm{d}t$. Using the substitution $u = st$ we find that

$$\int_0^\infty \frac{t^{x-1}}{st+1}\,\mathrm{d}t = \int_0^\infty \frac{1}{u+1}\left(\frac{u}{s}\right)^{x-1}\frac{\mathrm{d}u}{s} = \frac{1}{s^x}\int_0^\infty \frac{u^{x-1}}{u+1}\,\mathrm{d}u = s^{-x}\phi(x).$$

Next, consider the integral $\displaystyle\int_0^\infty \frac{t^{x-1}}{t+s}\,\mathrm{d}t$. Using the substitution $u = \frac{t}{s}$ we obtain

$$\int_0^\infty \frac{t^{x-1}}{t+s}\,\mathrm{d}t = \int_0^\infty \frac{(su)^{x-1}}{s(u+1)}\,s\,\mathrm{d}u = s^{x-1}\int_0^\infty \frac{u^{x-1}}{u+1}\,\mathrm{d}u = s^{x-1}\phi(x).$$

Subtracting the first integral from the second gives

$$\begin{aligned}
\phi(x)(s^{x-1} - s^{-x}) &= \int_0^\infty \left(\frac{t^{x-1}}{t+s} - \frac{t^{x-1}}{st+1}\right)\mathrm{d}t \\
&= \int_0^\infty \frac{t^{x-1}(t-1)(s-1)}{(st+1)(t+s)}\,\mathrm{d}t \\
\implies \phi(x)\frac{s^{x-1} - s^{-x}}{s-1} &= \int_0^\infty \frac{t^{x-1}(t-1)}{(st+1)(t+s)}\,\mathrm{d}t. \qquad (4.1)
\end{aligned}$$

Now, consider

$$
\begin{aligned}
[\phi(x)]^2 &= \left(\int_0^\infty \frac{t^{x-1}}{1+t}\,\mathrm{d}t\right)\left(\int_0^\infty \frac{s^{x-1}}{1+s}\,\mathrm{d}s\right) \\
&= \int_0^\infty \int_0^\infty \frac{t^{x-1}s^{x-1}}{(1+t)(1+s)}\,\mathrm{d}t\,\mathrm{d}s \\
&= \int_0^\infty \frac{1}{1+s}s^{x-1}\int_0^\infty \frac{t^{x-1}}{1+t}\,\mathrm{d}t\,\mathrm{d}s \\
&= \int_0^\infty \frac{1}{1+s}\int_0^\infty \frac{t^{x-1}}{t+s}\,\mathrm{d}t\,\mathrm{d}s && \text{as } \int_0^\infty \frac{t^{x-1}}{t+s}\,\mathrm{d}t = s^{x-1}\phi(x) \\
&= \int_0^\infty \int_0^\infty \frac{t^{x-1}}{(1+s)(t+s)}\,\mathrm{d}s\,\mathrm{d}t && \text{changing the order of integration} \\
&= \int_0^\infty \frac{t^{x-1}}{t-1}\int_0^\infty \left(\frac{1}{1+s}-\frac{1}{t+s}\right)\mathrm{d}s\,\mathrm{d}t && \text{using partial fractions} \\
&= \int_0^\infty \frac{t^{x-1}}{t-1}\left[\log\left(\frac{1+s}{t+s}\right)\right]_{s=0}^\infty \mathrm{d}t \\
&= \int_0^\infty \frac{t^{x-1}}{t-1}\left(\lim_{s\to\infty}\left(\log\frac{1+1/s}{1+t/s}\right)-\log\frac{1}{t}\right)\mathrm{d}t \\
&= \int_0^\infty \frac{t^{x-1}\log t}{t-1}\,\mathrm{d}t.
\end{aligned}
$$

Integrating this with respect to x over $[1-y, y]$ gives

$$
\begin{aligned}
\int_{1-y}^{y}[\phi(x)]^2\,\mathrm{d}x &= \int_{1-y}^{y}\int_0^\infty \frac{t^{x-1}\log t}{t-1}\,\mathrm{d}t\,\mathrm{d}x \\
&= \int_0^\infty \frac{1}{t-1}\int_{1-y}^{y}(t^{x-1}\log t)\,\mathrm{d}x\,\mathrm{d}t && \text{changing the order of integration} \\
&= \int_0^\infty \frac{1}{t-1}\left[t^{x-1}\right]_{x=1-y}^{y}\mathrm{d}t \\
&= \int_0^\infty \frac{t^{y-1}-t^{-y}}{t-1}\,\mathrm{d}t.
\end{aligned}
$$

Changing t to s, x to t and y to x, this becomes

$$
\int_{1-x}^{x}[\phi(t)]^2\,\mathrm{d}t = \int_0^\infty \frac{s^{x-1}-s^{-x}}{s-1}\,\mathrm{d}s. \tag{4.2}
$$

Note that the integrand in the right-hand side of (4.2) resembles (4.1). So, take (4.1) and integrate both sides with respect to s over $[0, \infty)$:

$$
\begin{aligned}
\phi(x)\int_0^\infty \frac{s^{x-1}-s^{-x}}{s-1}\,\mathrm{d}s &= \int_0^\infty\int_0^\infty \frac{t^{x-1}(t-1)}{(st+1)(t+s)}\,\mathrm{d}t\,\mathrm{d}s \\
\implies \phi(x)\int_{1-x}^{x}[\phi(t)]^2\,\mathrm{d}t &= \int_0^\infty t^{x-1}(t-1)\int_0^\infty \frac{1}{(st+1)(t+s)}\,\mathrm{d}s\,\mathrm{d}t
\end{aligned}
$$

where we have used (4.2) and changed the order of integration. Now using partial fractions, we obtain

$$\begin{aligned}\phi(x)\int_{1-x}^{x}[\phi(t)]^2\,\mathrm{d}t &= \int_0^\infty \frac{t^{x-1}(t-1)}{t^2-1}\int_0^\infty\left(\frac{t}{st+1}-\frac{1}{t+s}\right)\mathrm{d}s\,\mathrm{d}t\\ &= \int_0^\infty \frac{t^{x-1}}{t+1}\left[\log\left(\frac{st+1}{t+s}\right)\right]_{s=0}^{\infty}\mathrm{d}t\\ &= \int_0^\infty \frac{t^{x-1}}{t+1}\left(\lim_{s\to\infty}\left(\log\frac{t+1/s}{1+t/s}\right)-\log\frac{1}{t}\right)\mathrm{d}t\\ &= \int_0^\infty \frac{t^{x-1}}{t+1}\left(\log t-\log\frac{1}{t}\right)\mathrm{d}t\\ &= 2\int_0^\infty \frac{t^{x-1}\log t}{t+1}\,\mathrm{d}t. \end{aligned}\tag{4.3}$$

Recall a standard result on differentiation under the integral sign, which says that for $f\colon [a,b]\times[h,k]\to\mathbb{R}$, if $\frac{\partial}{\partial x}f(t,x)$ is continuous, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^b f(t,x)\,\mathrm{d}t = \int_a^b \frac{\partial}{\partial x}f(t,x)\,\mathrm{d}t.$$

The result also holds for improper integrals. (For a proof of this result, see Rudin [**9**, thm. 9.42].) So, differentiating $\phi(x)=\displaystyle\int_0^\infty \frac{t^{x-1}}{1+t}\,\mathrm{d}t$, we obtain

$$\phi'(x)=\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty\frac{t^{x-1}}{1+t}\,\mathrm{d}t=\int_0^\infty\frac{\partial}{\partial x}\left(\frac{t^{x-1}}{1+t}\right)\mathrm{d}t=\int_0^\infty\frac{t^{x-1}\log t}{1+t}\,\mathrm{d}t.$$

Combining this with (4.3) yields

$$\phi(x)\int_{1-x}^{x}[\phi(t)]^2\,\mathrm{d}t = 2\phi'(x).\tag{4.4}$$

Now, since $\phi(x)=\Gamma(x)\Gamma(1-x)$, we have $\phi(x)=\phi(1-x)$ for all x. Hence $\phi'(x)=-\phi'(1-x)$, so $\phi'\left(\frac{1}{2}\right)=-\phi'\left(\frac{1}{2}\right)$ and hence $\phi'\left(\frac{1}{2}\right)=0$. Furthermore,

$$\int_{1-x}^{x}[\phi(t)]^2\,\mathrm{d}t=\int_{1-x}^{1/2}[\phi(t)]^2\,\mathrm{d}t+\int_{1/2}^{x}[\phi(t)]^2\,\mathrm{d}t.$$

Letting $u=1-t$ in the first integral on the right-hand side, we obtain

$$\int_{1-x}^{x}[\phi(t)]^2\,\mathrm{d}t=\int_{1/2}^{x}[\phi(1-u)]^2\,\mathrm{d}u+\int_{1/2}^{x}[\phi(t)]^2\,\mathrm{d}t=2\int_{1/2}^{x}[\phi(t)]^2\,\mathrm{d}t$$

since $\phi(1-u)=\phi(u)$. Combining this with (4.4) yields

$$\phi'(x)=\phi(x)\int_{1/2}^{x}[\phi(t)]^2\,\mathrm{d}t.$$

Differentiating both sides of this identity yields

$$\phi''(x)=\phi(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{1/2}^{x}[\phi(t)]^2\,\mathrm{d}t\right)+\phi'(x)\int_{1/2}^{x}[\phi(t)]^2\,\mathrm{d}t$$

Now $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\displaystyle\int_{1/2}^{x}[\phi(t)]^2\,\mathrm{d}t\right) = [\phi(x)]^2$, and $\displaystyle\int_{1/2}^{x}[\phi(t)]^2\,\mathrm{d}t = \frac{\phi'(x)}{\phi(x)}$, so

$$\phi(x)\phi''(x) = [\phi(x)]^4 + [\phi'(x)]^2. \tag{4.5}$$

This is our differential equation. However, this is only half the problem; we must now solve this differential equation, with the conditions that $\phi\left(\frac{1}{2}\right) = \left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \pi$ and $\phi'\left(\frac{1}{2}\right) = 0$. Rearranging (4.5) we obtain

$$\phi^2 = \frac{1}{\phi}\phi'' - \frac{1}{\phi^2}\left(\phi'\right)^2.$$

Thinking of $\phi' = \dfrac{\mathrm{d}\phi}{\mathrm{d}x}$ as a separate variable, we obtain $\phi'' = \dfrac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = \dfrac{\mathrm{d}\phi'}{\mathrm{d}x} = \dfrac{\mathrm{d}\phi'}{\mathrm{d}\phi}\cdot\dfrac{\mathrm{d}\phi}{\mathrm{d}x} = \phi'\dfrac{\mathrm{d}\phi'}{\mathrm{d}\phi}$ and hence

$$\phi^2 = \frac{\phi'}{\phi}\frac{\mathrm{d}\phi'}{\mathrm{d}\phi} - \frac{(\phi')^2}{\phi^2}.$$

Now, consider that

$$\frac{\mathrm{d}\left((\phi')^2\right)}{\mathrm{d}\left(\phi^2\right)} = \frac{\mathrm{d}\left((\phi')^2\right)}{\mathrm{d}\phi'}\frac{\mathrm{d}\phi'}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}\left(\phi^2\right)} = 2\phi'\frac{\mathrm{d}\phi'}{\mathrm{d}\phi}\frac{1}{\frac{\mathrm{d}(\phi^2)}{\mathrm{d}\phi}} = \frac{\phi'}{\phi}\frac{\mathrm{d}\phi'}{\mathrm{d}\phi}.$$

Hence

$$\frac{\mathrm{d}\left((\phi')^2\right)}{\mathrm{d}\left(\phi^2\right)} = \phi^2 + \frac{(\phi')^2}{\phi^2}.$$

Multiplying through by ϕ^2 and rearranging yields

$$\phi^4 = \phi^2\frac{\mathrm{d}\left((\phi')^2\right)}{\mathrm{d}\left(\phi^2\right)} - (\phi')^2.$$

Dividing through by ϕ^4, we see that the right-hand side is the derivative of a quotient:

$$1 = \frac{\phi^2\frac{\mathrm{d}\left((\phi')^2\right)}{\mathrm{d}(\phi^2)} - (\phi')^2}{\phi^4} = \frac{\mathrm{d}}{\mathrm{d}\left(\phi^2\right)}\left(\frac{(\phi')^2}{\phi^2}\right).$$

Integrating the first and last expressions with respect to ϕ^2 yields $\phi^2 + k = \frac{(\phi')^2}{\phi^2}$. Substituing $\phi\left(\frac{1}{2}\right) = \pi$ and $\phi'\left(\frac{1}{2}\right) = 0$ yields $k = -\pi^2$ and hence

$$\phi^2 - \pi^2 = \frac{(\phi')^2}{\phi^2}.$$

Rearranging this yields

$$\frac{\phi'}{\phi^2\sqrt{1-\frac{\pi^2}{\phi^2}}} = \pm 1.$$

Let $\psi = \dfrac{\pi}{\phi}$, then $\dfrac{\mathrm{d}\psi}{\mathrm{d}x} = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\pi}{\phi}\right) = -\dfrac{\pi\phi'}{\phi^2}$, and we obtain

$$\frac{1}{\pi\sqrt{1-\psi^2}}\cdot\frac{\mathrm{d}\psi}{\mathrm{d}x} = \pm 1.$$

Integrating with respect to x gives

$$\begin{aligned}\frac{1}{\pi}\int\frac{\mathrm{d}\psi}{\sqrt{1-\psi^2}} &= \pm(x+c)\\ \implies \frac{1}{\pi}\arccos\psi &= \pm(x+c)\\ \implies \psi=\frac{\pi}{\phi} &= \cos(\pm\pi(x+c)) = \cos\pi(x+c)\\ \implies \phi(x) &= \frac{\pi}{\cos\pi(x+c)}\end{aligned}$$

where c is an arbitary constant of integration. Now

$$\phi\left(\tfrac{1}{2}\right)=\pi=\frac{\pi}{\cos(c\pi+\pi/2)}$$

and hence $\cos(c\pi+\pi/2)=-\sin(c\pi)=1$ and so $\cos(c\pi)=0$. Hence

$$\cos\pi(x+c)=\cos(\pi x)\cos(c\pi)-\sin(\pi x)\sin(c\pi)=\sin(\pi x).$$

Hence, finally,

$$\phi(x)=\Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin(\pi x)}. \qquad \square$$

5 Weierstrass' Product

We have now have two answers to the question why do we use the Gamma function: it is the only extension of the factorials which is log-convex, and it is most useful in calculuating a variety of integrals. Finally, we now consider another form for the Gamma function with even more applications.

Theorem 5.1 (Weierstrass' form of $\Gamma(x)$)**.** *For all $x\in(0,\infty)$,*

$$\Gamma(x)=\frac{1}{xe^{\gamma x}}\prod_{k=1}^{\infty}\frac{e^{x/k}}{1+\frac{x}{k}} \tag{5.1}$$

where $\gamma=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log n\right)$ is Euler's constant.

Proof. We take the right-hand side of the expression in (5.1) and show that it equals the right-hand side of (2.3) in corollary 2.9:

$$\begin{aligned}\frac{1}{xe^{\gamma x}}\prod_{n=1}^{\infty}\frac{e^{x/n}}{1+\frac{x}{n}} &= \lim_{n\to\infty}\frac{1}{x}\exp\left(-x\left[1+\tfrac{1}{2}+\cdots+\tfrac{1}{n}-\log n\right]\right)\prod_{k=1}^{n}\frac{\exp\frac{x}{k}}{\left(\frac{x+k}{k}\right)}\\ &= \lim_{n\to\infty}\frac{1}{x}\prod_{k=1}^{n}\exp-\frac{x}{k}\cdot\exp(x\log n)\cdot\prod_{k=1}^{n}\exp\frac{x}{k}\cdot\prod_{k=1}^{n}\frac{k}{x+k}\\ &= \lim_{n\to\infty}\frac{n!n^x}{x(x+1)\dots(x+n)}.\end{aligned} \qquad \square$$

This is one of three equivalent definitions given *a priori* in Burkill and Burkill [**5**], along with Euler's integral (1.2) and Gauss' product (2.3). In fact,

$$\Gamma(1+x)=x\Gamma(x)=\frac{1}{e^{\gamma x}}\prod_{k=1}^{\infty}\frac{e^{x/k}}{1+\frac{x}{k}}$$

is the form in which it is stated there. Using this form, we can derive an expression for $\sin\pi x$ as an infinite product:

Proposition 5.2. *For* $0 < x < 1$, $\sin \pi x = \pi x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right)$.

Proof. By proposition 4.1,

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

for $0 < x < 1$. Multiplying both sides by x, we get

$$\Gamma(1+x)\Gamma(1-x) = x\Gamma(x)\Gamma(1-x) = \frac{\pi x}{\sin \pi x}.$$

By theorem 5.1, we have

$$\Gamma(1+x) = \frac{1}{e^{\gamma x}} \prod_{k=1}^{\infty} \frac{e^{x/k}}{1 + \frac{x}{k}} \qquad \text{and} \qquad \Gamma(1-x) = \frac{1}{e^{-\gamma x}} \prod_{k=1}^{\infty} \frac{e^{-x/k}}{1 - \frac{x}{k}}.$$

Multiplying these gives

$$\begin{aligned} \Gamma(1+x)\Gamma(1-x) &= \frac{1}{e^{\gamma x}} \frac{1}{e^{-\gamma x}} \prod_{k=1}^{\infty} \frac{e^{x/k}}{1 + \frac{x}{k}} \cdot \prod_{k=1}^{\infty} \frac{e^{-x/k}}{1 - \frac{x}{k}} \\ &= \prod_{k=1}^{\infty} \frac{e^{x/k}}{1 + \frac{x}{k}} \cdot \frac{e^{-x/k}}{1 - \frac{x}{k}} = \frac{1}{\prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2}\right)}. \end{aligned}$$

Hence

$$\frac{\pi x}{\sin \pi x} = \frac{1}{\prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2}\right)}$$

and rearranging gives

$$\sin \pi x = \pi x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right). \qquad \square$$

This is fascinating: we started out with all kinds of integrals, and we have ended up with a seemingly unrelated product for $\sin \pi x$. This is but one such result: from this we can go on to derive similar expressions for the other trigonometric functions, which prove useful in many applications to elliptic functions and other special functions used in mathematical physics and differential equations.

So far we have been constrained to considering $\Gamma(x)$ only for $0 < x < \infty$. As is often useful in the study of special functions, we consider what happens if we try and extend the Gamma function to values other than $0 < x < \infty$. This is done by a process called *analytic continuation.*

Analytic continuation is actually a quite familiar process: in essence, we have already done this in extending the factorial function to the Gamma function. Another example would be to take the power series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. The sum on the left-hand side converges only for $-1 < x < 1$, but the right-hand side is well-defined for all $x \neq 1$. Hence by using this formula, we have extended the function $\sum_{n=0}^{\infty} x^n$ from domain $(-1, 1)$ to domain $\mathbb{R} \setminus \{1\}$.

Our original definition of the Gamma function in (1.2) relies on an integral which is only defined for $0 < x < \infty$. However, we have proved in (2.3) and (5.1) that

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} \, \mathrm{d}t = \lim_{n \to \infty} \frac{n! n^z}{z(z+1)\ldots(z+n)} = \frac{1}{z e^{\gamma z}} \prod_{k=1}^{\infty} \frac{e^{z/k}}{1 + \frac{z}{k}}$$

for all $0 < z < \infty$. The second expression for $\Gamma(z)$, Gauss' product, is tantalising, but defining n^z for z outside $0 < z < \infty$ is tricky; we will not take this path. However, Weierstrass' product relies only on elementary operations and the exponential function e^z, which is well-defined for any complex number z. The denominator is zero at $z = 0, -1, -2, \ldots$, due to the presence of z and $(1 + \frac{z}{k})$, but at all other complex numbers z, Weierstrass' product is well-defined.

However, the Gamma function on the complex plane is best stuided by looking at its reciprocal:

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{k=1}^{\infty} \frac{1 + \frac{z}{k}}{e^{z/k}}.$$

Now the only factors on the bottom line are exponentials. Now, since $\exp(z)\exp(w) = \exp(z+w)$ for all complex numbers z and w, we have in particular that $\exp(z)\exp(-z) = \exp(0) = 1$. Hence $\exp(z) \neq 0$ for any complex number z. This means that $\frac{1}{\Gamma(z)}$ is well-defined for any complex number z.

Moreover, the true beauty of $\frac{1}{\Gamma} \colon \mathbb{C} \to \mathbb{C}$ is that not only is it defined on all of $\mathbb{C}$, it is in fact *holomorphic* on all of $\mathbb{C}$, i.e. it is an entire function!

Theorem 5.3. *$f \colon \mathbb{C} \to \mathbb{C}$ given by*

$$f(z) := \frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{k=1}^{\infty} \frac{1 + \frac{z}{k}}{e^{z/k}}$$

is holomorphic on all of $\mathbb{C}$.

Since $\frac{1}{\Gamma(z)} = 0$ for $z = 0, -1, -2, \ldots$, this implies that $\Gamma(z)$ is meromorphic, i.e. holomorphic except at countably many poles. Recall that a holomorphic function on a connected region $\Omega \subset \mathbb{C}$ is necessarily analytic – i.e. it equals its Taylor series around any point – and is hence *infinitely* differentiable. This means that $\Gamma(z)$ possesses derivatives of all orders at every point in $\mathbb{C} \setminus \{0, -1, -2, \ldots\}$.

We can also extend theorem 4.1 to any $x \in \mathbb{C} \setminus \mathbb{Z}$; a proof of this may be found in Andrews et al. [**3**, thm. 1.2.1]. From this we find that proposition 5.2 holds for any $x \in \mathbb{C}$; furthermore, using the derivatives of Γ, we can obtain even more relations in this spirit.

The proof of this beautiful result is unfortunately beyond the scope of this essay. Essentially, it is sufficient to show that the product converges, i.e. that the sequence

$$f_n(x) = ze^{\gamma z} \prod_{k=1}^{n} \frac{1 + \frac{z}{k}}{e^{z/k}}$$

converges, and that this convergence is uniform on any compact subset of $\mathbb{C}$; see Rudin [**10**, thm. 15.6]. Then, since each f_n is the product of finitely many holomorphic functions, each f_n is holomorphic; it is then a standard result that if $f_n \to f$ uniformly on any compact subset of $\mathbb{C}$ and each f_n is holomorphic, then f is holomorphic; see Rudin [**10**, thm. 10.28]. Alternatively, but somewhat more opaquely, one may appeal to the factor theorem of Weierstrass; see Burkill and Burkill [**5**, thm. 14.51] or Ablowitz and Fokas [**1**, thm. 3.6.4].

We have thus discovered at least three reasons for considering the Gamma function as *the* extension of the factorials: it is log-convex on $(0, \infty)$, it is meromorphic on $\mathbb{C} \setminus \{0, -1, -2, \ldots\}$, and it gives rise to a huge theory of special functions, as alluded to by proposition 5.2. However, this is but a taste of the power of the Gamma function. The article by Davis [**7**] provides much more of the history of the Gamma function and motivation for its use, while Artin introduces many aspects of the Gamma function not found here in his lucid monograph [**4**].

Even more about the Gamma and Beta functions, including the links between the Gamma function and the Riemann zeta function, leading on to a full development of the theory of hypergeometric functions, can be found in the excellent *Special Functions* by Andrews et al. [**3**]. The preface to that book sums up the Gamma function perfectly: "[Euler himself] could not have foreseen the extent of its importance in mathematics".

References

1. Mark J. Ablowitz and Athanassios S. Fokas. *Complex Variables: Introduction and Applications.* Cambridge Texts in Applied Mathematics. Cambridge University Press, second edition, 2003.

2. Milton Abramowitz and Irene A. Stegun, editors. *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables.* Dover Publications, Inc., New York, tenth edition, December 1972.

3. George E. Andrews, Richard Askey, and Ranjan Roy. *Special Functions*, volume 71 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, 1999.

4. Emil Artin. *The Gamma Function.* Holt, Rinehart and Winston, Inc., 1964. Translated by Michael Bulter.

5. J. C. Burkill and H. Burkill. *A Second Course in Mathematical Analysis.* Cambridge University Press, 1970.

6. Marek Capiński and Ekkehard Kopp. *Measure, Integral and Probability.* Springer-Verlag London Ltd., second edition, 2004.

7. Philip J. Davis. Leonhard Euler's Integral: A Historical Profile of the Gamma Function. *The American Mathematical Monthly*, 66(10):849–869, December 1959. In Memoriam: Milton Abramowitz.

8. R. Dedekind. Über ein Eulersches Integral. *Journal für die reine und angewandte Mathematik*, 1853.

9. Walter Rudin. *Principles of Mathematical Analysis.* McGraw-Hill International Editions, third edition, 1976.

10. Walter Rudin. *Real and Complex Analysis.* McGraw-Hill International Editions, third edition, 1987.

11. Thomas Schmelzer and Lloyd N. Trefethen. Computing the Gamma Function Using Contour Integrals and Rational Approximations. *SIAM J. Numer. Anal.*, to appear. URL `http://web.comlab.ox.ac.uk/oucl/work/nick.trefethen/gamma.pdf`. [Online; accessed 31-December-2006].

12. Peter Walker. *Examples and Theorems in Analysis.* Springer-Verlag London Limited, 2004.