What is a Cardinal Number?

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Introduction

The starting point of this essay is chapter 8 of the lecture notes for Foundations. It is written as though intended for a first year student who has completed that course and is not satisfied with only understanding cardinality as it is explained there. This essay explains what a cardinal number is, with the emphasis on the concepts since the technical details are quite tricky. From time to time, the Foundations lecture notes are used as a you-should-know-this-already reference. Referencing is perhaps a little careless sometimes, with maybe a whole paragraph mainly drawn from a text cited halfway through, and occasionally a direct ciataion may be missing altogether. Nevertheless, most of the material in the essay can be found among the references, with a few illustrative examples invented purely for this essay.

Well-Ordering

You will all know what an *equivalence relation* is. It is a relation satisfying three conditions; reflexivity, symmetry and transitivity [F, p88]. To begin with, we need to discuss something similar - an order relation, mentioned in the same source but not defined formally.

Definition [D&R, p51] **:** An *order relation* R on a set X is a relation satisfying:-

(1) *reflexivity*: $\forall a \in X$, aRa.

(2) *antisymmetry*: $\forall a,b \in X$, (aRb and bRa) $\Rightarrow$ a=b.

(3) *transitivity*: $\forall a,b,c \in X$, (aRb and bRc) $\Rightarrow$ aRc.

When defining antisymmetry, we mean that a and b are the same thing when we write a=b. We haven't picked an equivalence relation out of thin air. The usual symbol for an order relation is not R, it tends to be called $\leq$. It seems to be conventional to deal with the concept represented by $\leq$, instead of the one denoted $<$. The latter is called a *strict order relation*, and apart from being transitive, it is *irreflexive* ($a \not< a$) and *asymmetric* ($a < b \Rightarrow b \not< a$). We don't need to worry about that here.

The order relations we shall be talking about are total orderings. An order relation $\leq$ defined on a set X is a *total order* [H, p54] if it is defined on the whole of the set of pairs of elements from X. That is, if for any a and b in X, $a \leq b$ or $b \leq a$. This is an important definition to make, as not all order relations are total orderings. The general order relation is called a *partial order*, and for many partial orders, there are pairs of elements in the set which simply cannot be compared with each other.

Example: An example of a partial order that is not total is when the set X is the set of intervals of the real line, and the order relation is set inclusion $\subseteq$. It is easy to check that $\subseteq$ is an order relation, it satisfies the definition. It also isn't hard to find a pair of intervals, such as (0,2) and (1,3), with neither contained within the other. Under this partial order, the two intervals are incomparable.

Luckily, we shall not have to deal with such unfinished orderings. In fact, we shall be examining a special class of total orderings which are called well orderings. Those who remember past courses will recall this from Foundations:

"**The Well-Ordering Principle:** Every non-empty subset of $\mathbb{N}$ has a least element."[F, p5]

It's called the wellordering principle (of the natural numbers) because the natural numbers under their obvious ordering is an example of a well-ordering. The strict definition is as follows:

Definition [Ca, p38] **:** A total ordering on a set X is called a *well-ordering* if every nonempty subset of X has a least element. By *least element* we mean an element a in the subset such that, for any element b in the subset, $a \leq b$.

Unfortunately, that may be a little difficult to imagine. It might help to give an example of something that is not a well ordering:

$$..., 4, 3, 2, 1$$

Here, I'm labelling the elements of the set with natural numbers, and the order relation is defined by $a \leq b$ meaning 'a is written to the left of b'. If you like, you can imagine replacing each comma with a $\leq$ sign. This is clearly not a well ordering. The whole set does not have a least element. If you find the definition of wellordering as it stands a bit hard to get your head around, you can think of a well ordering as an ordering without any infinitely decreasing sequences like the one above. It means more or less the same thing.

Beware though, some peculiar-looking things are wellorderings. Like this, for instance:

$$1, 3, 5, ..., 2, 4, 6, ...$$

Infinitely increasing sequences shouldn't surprise you. What may surprise you is elements on top of an infinite sequence, and infinite sequences on top of infinite sequences. This still makes sense as an order relation. Any odd number is less than or equal to any even number. And it is a wellordering. If a nonempty subset contains any odd numbers, the smallest element will be the first odd number; otherwise, the least element will be the first even number. Wellorderings like this are going to appear in the discussion about ordinal numbers. Hopefully, you can cope.

Ordinal Numbers

Right. So, what have we come up with this thing called wellordering for? Well, we are trying to find out what a cardinal is, after all. We can't easily count the elements of a arbitrary set, so we started thinking of cardinals as equivalence classes under bijection. That won't do though, as we're looking for a definite cardinal number. What we want is a way of counting the elements of a set. That's why we've introduced wellordering; for a well-ordered set we will be able to label each element with it's own number, which tells you where the element is in the ordering. That's more or less what we mean by an *ordinal number*, although I'm going to put off formally defining it for a while.

Well, what are we on about when we talk about counting the elements of a set in this way? Now now, be patient. It's easy to see how you can begin to count the elements of a set. You start by labelling the first elements of a set with natural numbers. But what about those pesky infinite chains in the middle of some wellorderings? Clearly we will exhaust the natural numbers on the first of those. And what then, how are we to label the rest? Can it even be done in a meaningful way? Calm down, you're getting me worried.

You can always label an element of one of these wellordered sets in a meaningful way. Consider the subset consisting of the elements which you haven't labelled yet. If that subset's empty, you're fine. Otherwise, being a nonempty subset of a well ordered set, it will have a least member. That member should get the next labelling (ordinal number) after all the ones we've had so far. It seems, therefore, that you can always find a next ordinal when you have all the ordinals that come before it. Does that sound at all familiar? Possibly not, but take a look at page 6 of the Foundations lecture notes. There you will find that the Principle of Induction is introduced, and it is proven with the Well-Ordering Principle, using an argument a lot like the informal one we've just used. But the Principle of Induction isn't quite the same - it only works on the natural numbers, and it involves considering only the number which came immediately before a given natural number, not all the numbers which came before. However, a little later in the notes, at the start of the next chapter, you will find this:

"Here is a slightly different Principle of Induction:

Principle of Induction II Suppose that $T \subseteq \mathbb{N}$ and that

1. $0 \in T$, and

2. for every natural number n, if $1, 2, ..., n-1$ are all in T, then $n \in T$.

Then T is all of $\mathbb{N}$."

Now that's more like it, that sounds like what we're doing. Here we consider all the numbers before the relevant one, and make our conclusions from that. So it sounds as though you can perform some sort of induction on the ordinals, and that it works slightly differently from the one we're used to. This process is called transfinite induction. It's given in [C, p40] as

"(Principle of transfinite induction) If a set A is well-ordered, $B \subset A$, and for every $x \in A$ the set B satisfies the condition

$$O(x) \subset B \Rightarrow x \in B,$$

then B=A."

In that text, O(x) means the set of elements before x. What I'm trying to say here is that the ordinals can be defined in terms of earlier ordinals, by a kind of transfinite induction. I still won't give the proper definition yet, but this should help you imagine what they are. Imagine a wellordered set before you, and you want to label the elements of the set with ordinals. Here, I'll write it out, replacing the elements of the set with Os.

$$O, O, O, ...O, O, O, ...O, O, ..O, O, ...O, O, O, ...$$

That's as clear as mud. I'm trying to convey an infinite sequence of infinite sequences, with an infinite sequence on top, just to illustrate how crazy these things can get. They go a long way beyond there, though. Regardless of that, we're trying to label these Os. You have to label the first element with zero, not one; you'll see why when we come to the definition. Now you have to label the rest. Notice that each ordinal has an ordinal that comes straight after it: these are called *successor ordinals*. For each of these successor ordinals, i.e. within an infinite sequence, induction could get by the way we are familiar with it; you could name each ordinal from the previous one and name all the ordinals in the sequence. But not all ordinals have an ordinal that came straight before; they are the ones at the start of a sequence, which are on top of a sequence. Those are called *limit ordinals*, and for that you need the transfinite induction, which 'gathers up' all the ordinals so far, and you can name the ordinal from that. Of course, this won't make sense without the definition, but isn't it about time I did something decent and named a few of these ordinals?[H, Ch19,20]

The first ordinal will be 0. The next one after that will be 1, after that 2, then 3, and all the natural numbers. I'd probably already made it clear that the first infinite sequence consists of the natural numbers. Then comes the ordinal after that, at the start of the next sequence. It is the first transfinite ordinal, and it will correspond with the natural numbers in some way. It's called ω. Some texts actually use the symbol ω in place of $\mathbb{N}$. The next one in the second sequence is called $\omega + 1$, followed by $\omega + 2$, $\omega + 3$ and so on. Presumably, the limit ordinal of that is called something like $\omega + \omega$. In fact, it's name is $\omega 2$ (not 2ω, that means something else). The next limit ordinal after that is called $\omega 3$, and later there is $\omega 4$, $\omega 5$ etc. At the end of an infinite sequence of infinite sequences is an ordinal like $\omega\omega$, whose real name is ω^2. And an infinite sequence of infinite sequences of infinite sequences has an associated ordinal called ω^3. I'd better stop describing these sequences now. Because you can carry on finding ω^4, ω^5 until you reach ω^ω. Later still you find ω^{ω^ω}. And $\omega^{\omega^{\omega^\omega}}$, $\omega^{\omega^{\omega^{\omega^\omega}}}$ ...is that still readable? Sources disagree on the name of the ordinal corresponding with an infinite tower of omegas. Halmos calls it ϵ_0. Of course there's no reason to stop there, you can go as far as you like with ordinals. Why not go as far as $\epsilon_0 + \epsilon_0 = \epsilon_0 2$, or $\epsilon_0 \epsilon_0 = \epsilon_0^2$? To infinity and beyond - $\epsilon_0^{\epsilon_0}$, $\epsilon_0^{\epsilon_0^{\epsilon_0}}$, and I think an infinite tower of those must be called ϵ_1, although none of the books get quite as carried away as that. I bet there is an ϵ_1 though, and an ϵ_2, ϵ_3... and just in case I haven't been sadistic enough, those ordinals are all countable.

I bet you're wishing I hadn't started naming them now. Regardless, there is one more way of thinking about ordinals which I'd like to explain before actually telling you what they are. You can think of ordinals as classifying types of wellordering. I've already implied that in the above paragraph, at the point when we mention ω^3. Just picture that wellordered set in front of you, with however many infinite chains it has. We want to come up with an ordinal number that describes the way the set is well ordered. There's a natural way to do that - just label the elements of the set with ordinals in the way we've been talking about so far, and when that's done, there'll be a first ordinal that wasn't used to label an element, which would have been the next ordinal used had there been any more elements in the set. That ordinal number represents the well-ordering itself. Once again we appear to be dealing with a mysterious connection between an ordinal number and the set of earlier ordinals. The wellordering defined by an ordinal number is the wellordering on the set of previous ordinals. Putting that aside for a little longer, we can now see some sense behind the earlier claim that 'an infinite sequence of infinite sequences of infinite sequences has an associated ordinal called ω^3'. We can also assign the ordinal $\omega 2$ to that first example of a wellordering, with odd and even numbers in different sequences, and to that well ordered set which had its elements replaced by the letter O, we can give the ordinal $\omega^2 + \omega$. Can this really be done for any well-ordered set? Yes it can; most of the references state and prove a theorem to the effect of 'any well-ordered set is *order-isomorphic* to a unique initial segment of ordinal numbers'. 'Initial segment' should be self-explanatory. The term 'order-isomorphic' isn't hard to define, but we needn't be pedantic - it just means we're relabelling the elements of the set with ordinals. The proof in [Ca p39ff] is accessible but takes some pages and would not be that illuminating to sketch out here. The proofs in the other references tend to be concise but require an axiomatic approach to set theory. This would make decent essay material, but there isn't time for it here. What all of the proofs need though, is the definition of ordinals, something we really can't procrastintae about for any longer.

Of course, I've given so many hints that you may well have guessed by now what the definition is. In case you haven't though, we'll firstly explain what the natural numbers are. Many familiar number systems are best conceived algebraically. For example the rationals are obtained by adding numbers to the set of integers until the result is a field. The natural numbers though, need to be thought of as sets. It seems natural for the cardinality of a set which is a natural number to be that natural number. Since in set theory we need to think of zero as a natural number, that sorts out the very first number.

$$0 := \emptyset$$

Now we need to define the next natural number. It should contain one element, which presumably needs to be a set as well. Hmm, we only have one set so far.

$$1 := \{\emptyset\} = \{0\}$$

Ah, is that how you get from one natural number to the next? The same reasoning leaves us without much choice for the next number.

$$2 := \{\emptyset, \{\emptyset\}\} = \{0, 1\}$$

Or the one after that.

$$3 := \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \{0, 1, 2\}$$

And so the natural numbers continue. If you haven't guessed what ordinals are by now, you'll be kicking yourself in a minute. Of course, this definition of the natural numbers is a slightly unsettling concept. We seem to be building numbers out of nothing. More than one of the references point out how disturbing it seems that, given natural numbers a and b with a less than b, not only is a a member of b, but a is a subset of b as well. However, if you can put up with that, then you should be ready for the following. We've been saying again and again that an ordinal number has something to do with the ordinals that came before it. Actually, what was meant all along is that an ordinal number *is* the set of all ordinal numbers that preceded it! Since that's one of our key concepts, we'd better give a formal definition. Here's the one from [Z p182]:

"A set X is called an **ordinal (number)** if $\in_X$ well-orders X and for each x in X, $x = sect_{\in_X}(x)$"

The english translation is probably clearer though. This explains many things that have appeared so far. It is now clear that the first ordinals are the natural numbers, which satisfy the definition. Including zero, which is indeed the set of its predecessors. The first transfinite ordinal must be the set of all finite ordinals, so

$$\omega = \{0, 1, 2, 3, ...\}$$

That's why $\mathbb{N}$ may be replaced with ω, as the symbols do mean the same thing. And ω^3 is a set of ordinals which, under their natural wellordering, form an infinite sequence of infinite sequences of infinite sequences. There is, strictly speaking, no set of all ordinal numbers. If you had such a set, then that itself would be an ordinal number greater than all ordinal numbers. However, the class of all ordinals is well ordered by its natural order relation. There are two choices for how you define this relation, one may say that $a < b$ if $a \in b$ or if $a \subset b$. Of course, the explanation about ordinals involving transfinite induction seems a trifle naive now.[1]

Well now we know what ordinals are, is there anything else we can do with them? You can define an addition operation and a multiplication operation on the class of ordinals, and these operations explain some of the notation for the names of individual ordinals. When considering these operations, the ordinals are best thought of as classifying wellordered sets. The definitions of these ordinal operations look elegant when written in symbols, but in the time taken to explain the symbols, the operations could be explained in plain english. Therefore we shall use the intuitive explanations given in [H, Ch21]. Given ordinals a and b, imagine corresponding wellordered sets A and B. Then imagine putting B on top of A, such that any element of A is less than or equal to any element of set B. The ordinal which represents the resulting well ordered set is a+b. A few examples may be in order. $\omega + \omega$ These ordinals are both infinite sequences, so their sum should correspond with one infinite sequence on top of another. Thus $\omega + \omega = \omega 2$, which makes sense. $1 + \omega$ This is, in effect, putting an infinite sequence on top of a single element, the result of which is an infinite sequence. Hence $1+\omega = \omega$. You'll notice that the sum $\omega+1$, adding an element on top of an infinite sequence, gives you the ordinal written as $\omega + 1$, which is just as well. However, that means ordinal addition isn't commutative. Neither is the multiplication operation, which is trickier to explain. By analogy with the natural numbers, as 3×5 can be explained as adding 3 to itself 5 times, so can the ordinal product $a \times b$ be explained as adding a to itself b times. For example $\omega \times \omega$ means that there is one infinite sequence for each natural number, and those sequences are placed on top of one another in the obvious way. The result is, of course, an infinite sequence of infinite sequences, or ω^2, so the notation has some sense behind it. $\omega \times 2$ By our explanation, this means the same thing as $\omega + \omega$, or $\omega 2$, which it ought to, really. But consider $2 \times \omega$. We have one ordered pair for each natural number, and we place them all atop one another. Lo and behold, we get a single infinite sequence, so $2 \times \omega = \omega$. An exponentiation operation can also be defined, which further explains the names of some of the ordinals, but apparently this is such a tricky concept that several of the references actually skip it. And with that, the best part of what can be done with ordinal numbers has been covered, and it's high time we got back on track.

[1]For a more thorough explanation, I'd recommend the Halmos.

Cardinal Numbers

We're trying to explain what a cardinal number is. It would be wise to recap what we already know from [F, Ch8]. In that chapter, the idea of a bijection is introduced, and it is explained that given two sets and a bijective mapping from one to the other, the sets must have the same cardinality. Following a number of results that relate only to bijections, the concept of countably infinite, or $|\mathbb{N}|$, is introduced, and the cardinality of several familiar sets is shown to be $|\mathbb{N}|$. It is then proven, using the famously ingenious Cantor's diagonalisation argument, that $|\mathbb{R}| > |\mathbb{N}|$. At that point it becomes neccesary to introduce a vague notion of cardinal number, and we are told that the number corresponding to $|\mathbb{N}|$ is named $\aleph_0$. An order relation is defined on the cardinals, that relation being $\aleph \leq \aleph'$ if, given two sets of those cardinalities, there is an injection from the first to the second. The following standard result is stated and proved.

Bernstein-Schröder Theorem If $|X| \leq |Y|$ and $|Y| \leq |X|$ then $|X| = |Y|$.

Strictly, that is part of the proof that the above is an order relation. Then the concept of the power set of X, $\mathcal{P}(X)$, which is the set of all subsets of X, is defined. It is then proven that $|X| < |\mathcal{P}(X)|$ for any set, which suggests that there are cardinals of arbitrarily large size. However at that point, save for an appendix, the topic is abruptly changed and a new chapter is begun.

We will have to explain what a cardinal number actually is. It will be obvious to you that this depends on the concept of ordinal number, otherwise we wouldn't have spent so much time on it. The only tricky thing is that the idea of ordinals depends so much on the idea of wellordering a set. This will require a little digression on the axiom of choice, a topic which could fill a whole essay by itself. Behind the heuristic explanations in this essay are the axioms of set theory. One of these, the Axiom of Choice, is still a little controvertial in spite of being largely tolerated nowadays. It says that, given a set of nonempty sets indexed by set A, one can choose a single element of each small set, one for each member of A. Reasons for suspicion concerning the axiom include that this assertion applies to any arbitrary set(s) regardless of any bizzare properties, that this axiom has been shown to imply some counterintuitive results (particularly the notorious Banach-Tarski Paradox[2]), and that the axiom is nonconstructive i.e. it claims that something can be done in any relevant situation but doesn't say how it should be done. Consequently, results which follow from the axiom of choice tend to do the same, including the one we'll resort to using. The reason that the axiom of choice is allowed to stay is that many standard results depend on it. The references include some of these proofs, such as; every vector space has a basis, Tychonoff's Theorem [C p55 and 58 respectively], and that there is a non-Lebesgue-measurable subset of the real numbers [Ca p122]. Evidently it is possible to do away with the axiom for defining cardinal numbers: Zuckerman's text discusses them in terms of a concept named Rank, but this is a bit tricky for an essay. It seems that, without the axiom of choice, there is a possibility

[2] Not discussed in detail in any of the references, but mentioned in [Ca,p122].

even more counterintuitive than Banach-Tarski - the existence of two sets of incomparable cardinality (no injection from one set to the other[Z p291]). This would surely be too much for us; from now on, we shall assume the axiom of choice and with it a corollary that is in fact equivalent to it, the Wellordering Principle [H p69] - that any set can be wellordered.

With this, it becomes much easier to explain what cardinals are than it was to explain what ordinals are. If every set can be well ordered, every set can be made to correspond with an ordinal number. The cardinality of that set will of course be the cardinality of the ordinal number. You will have noticed that many ordinals can have the same cardinality - most of the ordinals mentioned so far are countably infinite, they correspond with well orderings of countably infinite sets. Perhaps it isn't obvious whether there are any uncountable ordinals, besides the fact that they should correspond with wellordered uncountable sets. However, consider the set of all countable ordinals. Ordinals being what they are, that set is itself an ordinal. It can't be countable though, we've had all the countable ordinals already. This then is how ordinals of a greater cardinality can be obtained. It should be clear why an ordinal can't have a cardinality greater than a later ordinal. Given two ordinals, $a < b$, a must be a subset of b, giving a natural injection from a to b. This means that all the ordinals of a particular cardinality occur together on the 'ordinal line'. To define what a cardinal number is, it seems natural to choose an ordinal number from each set of ordinals of the right cardinality to represent those ordinals and the cardinality itself. Since sets of ordinals are wellordered, and that concept is characterised by least elements, it's a no-brainer how to define cardinals now.

Definition A *cardinal number* is an ordinal number such that all its elements have a cardinality less than the ordinal itself [H p100].

Texts also refer to these as *initial ordinals* when they are to be thought of as ordinal numbers rather than being the cardinality of a set. It can now be shown that each cardinal number has a least cardinal that is greater than it. Consider two initial ordinals, $a < b$, and consider the set of initial ordinals that are strictly between a and b. If it's empty, then b is the next cardinal after a. Otherwise, this forms a nonempty subset of the ordinal b, which is wellordered. Hence the set has a least element. It is not difficult to conceive what the next cardinal after a finite cardinal is. For infinite cardinals, we need names. Since every element of ω is finite, $\omega = \aleph_0$ must be the first infinite cardinal. The next one is called ω_1 when it is meant as an initial ordinal and $\aleph_1$ as a cardinal. This is followed by $\omega_2 = \aleph_2$, $\omega_3 = \aleph_3$ ... Here we have some infinite cardinals being indexed by natural numbers. Alarmingly, according to [Ca p125], the infinite cardinals are in fact indexed by ordinal numbers. Some of the implications of that are so mindboggling that it's probably best not to think about it.

At last, we've answered the main question of the essay. What else can we do with cardinals? Operations of addition, multiplication and exponentiation can be defined for cardinals as well as for ordinals. If a and b are cardinal numbers, a+b is defined as $|A \cup B|$, where A and B are disjoint sets of cardinalities a and b, and a×b is defined as $|A \times B|$. These operations are much nicer than their ordinal counterparts. In particular, they are both commutative. They are also a little trivial - for finite cardinals they coincide with the familiar operations on natural numbers. When one of a, b is infinite, $a + b = a \times b = max\{a, b\}$. [Ca p126] has the most readable proof. Cardinal exponentiation is a little more interesting, if a little less easy to imagine. a^b is defined to be the cardinality of the set of functions from B to A. Even if this a hard definition, cardinal exponentiation still satisfies some neat properties, such as $a^{b+c} = a^b a^c$, $(ab)^c = a^c b^c$ and $a^{bc} = (a^b)^c$ [H p96]. Perhaps the most important result about exponentiation though, is that for any cardinal a, $a < 2^a$. That will be a corollary of the next theorem. For now that we've introduced cardinal exponentiation, we will be able to give a couple of practical results whose proofs are not intimidatingly technical.

Theorem For any set X, $|\mathcal{P}(X)| = 2^{|X|}$.

Proof[Ca p128] We have a set of cardinality $|X|$, and for a set of cardinality 2, we can use {0,1}. We're looking for a bijection between $\mathcal{P}(X)$ and the set of functions from X to {0,1}. That's quite easy, given a function from X to {0,1}, map it to the subset which is the preimage of 1.

This tells us that, as an example, $\mathcal{P}(\mathbb{N}) = 2^{\aleph_0}$. The next theorem is probably the most practically useful one in the essay.

Theorem

$$|\mathbb{R}| = 2^{\aleph_0}$$

Proof[C p71]

(1) Define a map $f : \mathbb{R} \mapsto \mathcal{P}(\mathbb{Q}), f(r) = \{q \in \mathbb{Q}, q < r\}$.

Bear in mind that between any two distinct real numbers there is a rational number. This means that for any two distinct real numbers, $a < b$, there is a rational number that belongs to $f(b)$ but not to $f(a)$. So $f(a) \neq f(b)$ thus the mapping is an injection. Therefore $|\mathbb{R}| \leq 2^{\aleph_0}$.

(2) Let A be the set of sequences in {0,2}. Since A is the set of functions from $\mathbb{N}$ to {0,2}, it has cardinality $2^{\aleph_0}$. Define a map $g : A \mapsto \mathbb{R}, g(\{a_1, a_2, a_3, ...\}) = 0.a_1 a_2 a_3...$, where that is a ternary expansion, not a decimal one. This map is injective, and if you think about it, you'll see why: the only real numbers with non-unique ternary representations have a 1 in at least one of those representations. So $2^{\aleph_0} \leq |\mathbb{R}|$.

Hence, by the Bernstein-Schröder Theorem, $|\mathbb{R}| = 2^{\aleph_0}$.

Many conclusions may be drawn from this. The cardinality of any interval of the real line is $2^{\aleph_0}$, since a bijection between $\mathbb{R}$ and the interval may be found using the arctanh function. Using the above rule for cardinal multiplication, we find that $|\mathbb{R}^2| = |\mathbb{R} \times \mathbb{R}| = 2^{\aleph_0} \times 2^{\aleph_0} = 2^{\aleph_0}$. This also means, for example, that $|\mathbb{C}| = 2^{\aleph_0}$. Using a similar argument inductively (not tranfinite this time), it can be shown that $|\mathbb{R}^n| = 2^{\aleph_0}$. The cantor set also has cardinality $2^{\aleph_0}$, in fact the cantor set is the image of g in the above proof. It shall be left to the reader to play about with injections, surjections and cardinal arithmetic and discover for themselves the cardinalities of various infinite sets. Here though, is an empirical observation which cannot be referenced or proven: that all the infinite sets that one tends to come across in everyday mathematical practice seem to have cardinality $\aleph_0$ or $2^{\aleph_0}$. In fact one never seems to come across an infinite set that doesn't have a cardinality of $\aleph_0, 2^{\aleph_0}, \mathcal{P}(\mathbb{R}), \mathcal{P}(\mathcal{P}(\mathbb{R}))$... The power set operation, $2^{|X|}$, seems like such a nice way to reach a greater cardinal that it's natural to presume that $2^{\aleph_0} = \aleph_1$. This is known as the Continuum Hypothesis, and the generalised statement $2^{\aleph_a} = \aleph_{a+1}$ for ordinal a is called the Generalised Continuum Hypothesis. And here is where we are forced to end the essay on the same anticlimax with which more than one of the references ends. We know the Axiom of Choice cannot be proven from the other axioms of set theory and must be taken as an axiom in its own right. Similarly, as was proven in two stages by Gödel in 1940 and Cohen in 1963[F p74], the Continuum Hypothesis cannot be proven true or false even when the Axiom of Choice is accepted unconditionally. Hence the Continuum Hypothesis, or the statement 'the Continuum Hypothesis is false' must be added as an axiom of its own, and mathematicians do not take such leaps of faith lightly. Implications of the truth or falsity of the Continuum Hypothesis must be carefully analysed, and the Ciesielski text deals with this sort of thing towards the end. Time will tell what mathematicians' eventual opinion on the Continuum Hypothesis will be. Just in case none of this essay has been understood by the reader however, here is a little aphorism to remember. I may well not be the first person to have come up with this, considering how unoriginal it is, but just bear in mind that:

All infinities are infinite, but some are infinitely more than others.

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