# The Gamma Function

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In this essay I will examine the Gamma function $\Gamma(x)$ by looking into the main theory behind it and the useful properties that this effort reveals. Like many problems within mathematics, the reward for solving them is not just the result, but also the ability to apply the method to other problems. To best understand the Gamma function the notion of convexity must be introduced leading to the central theorem of the study of $\Gamma(x)$; Bohr-Mollerup. Even on it's own the Gamma function has far reaching links with a wide range of areas including probability, calculus and, most surprisingly, the Riemann Zeta function $\zeta(z)$.

## 1 Beginning With The Factorial

Before introducing the Gamma function it helps to first consider a different, but recognisable function $n!$.

$$n! := 1 \cdot 2 \cdot 3 \ldots (n-1) \cdot n \ \ \forall n \in \mathbb{N} \ \text{ and } 0! = 1$$

**Note.** $n! = n.(n-1)!$

The factorial is used quite commonly in mathematics, but the function is not defined on $\mathbb{R}$. It may be useful to extend $n!$ to the entire real line. A simple first step would be to carry across the previous formula where it is defined, but how best to define the other values in this extended factorial function? There are an infinite number of ways of doing this. Maybe you could take the floor function ($\lfloor x \rfloor$), or $x! = 0 \ \forall x \notin \mathbb{N}$. Still, it would be nice if it was continuous, or even differentiable. How should the points be joined? This is where Gamma comes in.

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$$

At a first glance no connection between the $n!$ and $\Gamma(x)$ can be seen, but

$$\Gamma(1) = \int_0^\infty e^{-t} t^0 dt = \int_0^\infty e^{-t} dt = [-e^{-t}]_0^\infty = 1$$

and using integration by parts

$$\Gamma(x+1) = \int_0^\infty e^{-t} t^x dt = [-e^{-t} t^x]_0^\infty + \int_0^\infty x e^{-t} t^{x-1} dt$$

$$= \int_0^\infty x e^{-t} t^{x-1} dt = x \int_0^\infty e^{-t} t^{x-1} dt = x\Gamma(x)$$

Then we have $\Gamma(x+1) = x\Gamma(x)$, this is very similar to $n! = n.(n-1)!$ . Using this fact we can compute $\Gamma(2) = \Gamma(1+1) = 1.\Gamma(1) = 1$ and $\Gamma(3) = \Gamma(2+1) = 2.\Gamma(2) = 1.2.\Gamma(1) = 2$. Applying this relationship with proof by induction we can calculate $\Gamma(x)\ \forall x \in \mathbb{N}$. In fact

$$\Gamma(x) = (x-1)! \quad \forall x \in \mathbb{N} \setminus \{0\}$$

It would be easy to make $\Gamma(x) = x!$ by adjusting the integral, but Legendre's definition didn't take this into account. A different notation was proposed by Gauss of $\Pi(x) = \Gamma(x-1)$. Making $\Pi(x) = x!$ . It is sometimes used but the other definition is much more well known. [6]

Now the link with $n!$ becomes clear. Very interestingly the relationship $\Gamma(x+1) = x\Gamma(x)$ applys for all $x$ in the reals. Going back to extending $n!$ , using this relationship means we only need to define the interval $(0,1]$ before it becomes defined everywhere else using this relationship. i.e.

$$f(x+1) = xf(x) \ \Rightarrow$$

$$y \in (0,1],\ n \in \mathbb{N} \qquad f(y+n) = (y+n-1)(y+n-2)\dots(y+1)yf(y)$$

Now taking this property in the other direction [2]

$$f(y) = \frac{1}{y(y+1)\dots(y+n-1)} f(y+n)$$

It must of course be assumed that $(y+n) \neq 0$. This means that the function is not defined for the negative integers or 0, but is defined everywhere else, using the relationship.

**Theorem 1** (Existence). $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$ *is defined everywhere except 0 and the negative integers*

*Proof.* First

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt = \int_0^1 e^{-t}t^{x-1}dt + \int_1^\infty e^{-t}t^{x-1}dt$$

Now $e^{-t} < 1$ for $t > 0 \Rightarrow$ For $x > 0$, $\delta \geqslant 0$

$$\int_\delta^1 e^{-t}t^{x-1}dt < \int_\delta^1 t^{x-1}dt = \frac{1-\delta^x}{x} < \frac{1}{x}$$

Decreasing $\delta$ makes the integral only larger in interval and hence $(e^{-t} > 0,\ t^{x-1} \geqslant 0)$ the value of the integral increases $\Rightarrow$

$$\lim_{\delta \to 0} \int_\delta^1 e^{-t}t^{x-1}dt = \int_0^1 e^{-t}t^{x-1}dt$$

This exists for $x > 0$, as it is increasing and bounded above by a constant when we fix an $x$.

Next, for $t > 0$, using the infinite expansion of $e^t$: For $n \in \mathbb{N}$

$$e^t < \frac{t^n}{n!} \ \Rightarrow \ e^{-t} < \frac{n!}{t^n} \ \Rightarrow \ e^{-t}t^{x-1} < \frac{n!}{t^{n+1-x}} = n!t^{x-1-n}$$

Now fix $x$ and set $n > x+1$, for example $n = \lceil x+2 \rceil$. Then for $\omega > 1$

$$\int_1^\omega e^{-t}t^{x-1}dt < \int_1^\omega n!t^{x-1-n}dt = \left[\frac{n!t^{x-n}}{x-n}\right]_1^\omega = \frac{n!}{x-n}(\omega^{x-n}-1) < \frac{n!}{n-x}$$

$$\lim_{\omega\to\infty}\int_1^\omega e^{-t}t^{x-1}dt = \int_1^\infty e^{-t}t^{x-1}dt$$

This is finite because, again, increasing $\omega$ makes the interval larger and the object is always positive. Therefore the integral increases in value. When $x$ is fixed it is bounded above by a constant.

Therefore $\int_0^\infty e^{-t}t^{x-1}dt = \Gamma(x)$ is finite for all $x > 0$. Now using the relation $\Gamma(x) = \frac{1}{x(x+1)\ldots(x+n-1)}\Gamma(x+n)$, as described above, $\Gamma(x)$ exists also for the negative half of the reals except for the negative integers and 0. $\square$

In modern day, computers calculating $\Gamma(x)$ for reasonable values use the interval $2 \leqslant x \leqslant 3$ as the resulting polynomial has the smallest coefficients. The formula $f(x+1) = xf(x)$ is then used as shown above. [5]

## 2 Convexity

Before moving further with $\Gamma$ it is necessary to introduce the notion of a convex function. The relation between the mathematical description of convex and the common notion of convex is very similar to the relation between continuity and "not taking the pen off the paper". They are similar and could be thought of in that way, but the mathematical definition requires more thought. Most of this section is taken from [2] and full proofs of the theorems I will state can be found there. The definition of a convex function is not the one usually given but is equivalent and I believe conceptually easier to visualise.

To start, define any function $f(x)$ defined on an open interval of the real line, for example $(l, u)$. Next, define

$$\varphi(a,b) = \frac{f(a)-f(b)}{a-b} = \varphi(b,a)$$

This is called the difference quotient and gives the straight line gradient between the 2 points $(a, f(a))$ and $(b, f(b))$, to put things simply. Next

$$\Psi(a,b,c) = \frac{\varphi(a,b)-\varphi(b,c)}{a-c} = \frac{(c-b)f(a)+(a-c)f(b)+(b-a)f(c)}{(a-b)(b-c)(c-a)}$$

**Note.** *The values of $a$, $b$ and $c$ can be permuted without changing $\Psi(a,b,c)$.*

**Definition 1.** *A function $f(x)$ is convex on an open interval when $\Psi(a,b,c) \geqslant 0$ for all $a$, $b$ and $c$ on that interval.*

Using the line gradient analogy, pick any three points on the graph and draw a line between the first and second point, and then between the second and third point. A function is convex if the gradient of the first line is always less than the gradient of the second line. So the gradient of the line between two points increases as you move them along the graph. A good example of convexity is $f(x) = x^2$. $f(x) = 0$ is also convex by the mathematical definition.

The other definition of convex is (for $x$ and $y$ in the interval and $0 < \lambda < 1$)

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda)f(y)$$

They are equivalent because, essentially, an extra point is created instead of using $\lambda$ and permuting them means one of a, b and c is always inbetween the other two.

**Theorem 2.** *The sum of two convex functions is convex.*

This is obvious from the second form of $\Psi(a, b, c)$.

**Theorem 3.** *An infinite sum of convex functions is also convex, as long as the sum exists and is finite.*

This takes the previous theorem further.

**Theorem 4.** *$f(x)$ is a convex function if, and only if, $f(x)$ has monotonically increasing one sided derivatives.*

As a consequence of this theorem a convex function must be continuous because one sided derivatives exist.

**Corollary 1.** *If $f(x)$ is twice differentiable, then $f''(x) \geqslant 0 \iff f(x)$ is convex.*

This corollary makes convex functions much easier to understand. Unfortunately convex functions are not necessarily twice differentiable.
Convexity is taken to a whole new degree by introducing log convexity.

**Definition 2.** *$f(x)$ is log convex if $f(x) > 0$ for all $x$ on an interval and $log(f(x))$ is convex.*

Log convexity is much stronger than convexity. In a way, it has to be exponentially more convex. A log convex function is always convex but a convex function is not always log convex. From here, similar to the previous theorems for convex functions, we have

**Theorem 5.** *The product of two log convex functions is log convex*

then

**Theorem 6.** *An infinite product of log convex functions is log convex as long as it exists and is finite.*

and

**Theorem 7.** *If $f(x) > 0$ and $f(x)f''(x) - (f'(x))^2 \geqslant 0$ then $f(x)$ is log convex.*

This is because the second derivative of $log(f(x))$ is $f(x)f''(x) - (f'(x))^2$.
Much less trivial and not immediately obvious is

**Theorem 8.** *The sum of two log convex functions is also log convex.*

I've omitted the proof but it is available in [2]. Returning to the Gamma function, the reason for defining convexity and log convexity was:

**Theorem 9.** $\Gamma(x)$ *is log convex ($x > 0$).*

*Proof.* First we must prove that $f(x,t) = e^{-t}t^{x-1}$ is log convex for all positive $t$.

$$log_e(e^{-t}t^{x-1}) = -t + (x-1)log(t)$$

Keeping $t$ as a constant and differentiating twice with respect to x gives $f''(x,t) = 0 \geqslant 0$ by Corollary 1. It's required that $t$ is positive so that $log(t)$ is finite. $e^{-t}t^{x-1}$ is therefore log convex.
Set

$$f_n(x) = h(f(x,0) + f(x,h) + f(x,2h) + \ldots + f(x,(n-1)h))$$

$h = \frac{b}{n}$ where 0 and $b$ are bounds of $t$. $f_n(x)$ is log convex because of Theorem 5.

$$lim_{n\to\infty} f_n(x) = \int_0^b f(x,t)dt$$

For the above equation to be convex the integral must be finite. If then $b \to \infty$ the integral is then equal to $\Gamma(x)$ and is therefore finite. Now, we have that $\Gamma$ is log convex. $\Gamma(x)$ is not log convex at 0 or any interval containing it. □

# 3 Bohr-Mollerup

Leading on from $\Gamma(x)$ being log convex comes a surprising theorem:

**Theorem 10** (Bohr Mollerup)**.** *There is only one function $f(x)$ which satisfies:*
*(1) $f(x+1) = xf(x)$*
*(2) For $f(x)$ is defined for all $x > 0$ and is log convex on this interval*
*(3) $f(1) = 1$*

*Proof.* [2]
(2) combined with a manipulation of the first expression of $\Psi(a,b,c)$ and the conditions $n \in \mathbb{N}$, $0 < x \leqslant 1$ leads to the dual inequality

$$\frac{logf(-1+n) - logf(n)}{(-1+n)-n} \leqslant \frac{logf(x+n) - logf(n)}{(x+n)-n} \leqslant \frac{logf(1+n) - logf(n)}{(1+n)-n}$$

cleaning this up and taking $f(n) = (n-1)!$ gives

$$log(n-1) \leqslant \frac{logf(x+n) - log(n-1)!}{x} \leqslant log\ n$$

$$log(n-1)^x(n-1)! \leqslant logf(x+n) \leqslant log\ n^x(n-1)!$$

$$(n-1)^x(n-1)! \leqslant f(x+n) \leqslant n^x(n-1)!$$

Next using (1) and (3) to give: $f(x+n) = (x+n-1)(x+n-2)\ldots(x+1)xf(x)$ and then dividing

$$\frac{(n-1)^x(n-1)!}{x(x+1)\ldots(x+n-1)} \leqslant f(x) \leqslant \frac{n^x(n-1)!}{x(x+1)\ldots(x+n-1)}$$

$$f(x) \leqslant \frac{n^x n!}{x(x+1)\dots(x+n)} \frac{x+n}{n}$$

Considering only the left inequality it holds for n as large as we want so we can replace n by n+1

$$\frac{n^x n!}{x(x+1)\dots(x+n)} \leqslant f(x) \leqslant \frac{n^x n!}{x(x+1)\dots(x+n)} \frac{x+n}{n}$$

Using the left inequality, and the right inequality divided by $\frac{x+n}{n}$, grants

$$f(x)\frac{n}{x+n} \leqslant \frac{n^x n!}{x(x+1)\dots(x+n)} \leqslant f(x)$$

Tending $n$ to infinity makes $f(x)$ alone a lower bound and we get

$$f(x) = \lim_{n\to\infty} \frac{n^x n!}{x(x+1)\dots(x+n)}$$

Therefore this single function defines $f(x)$ which satisfies the 3 conditions. □

Since $\Gamma(x)$ also fullfills the 3 conditions. Therefore $\Gamma(x) = f(x)$. But, more importantly,

$$\Gamma(x) = \lim_{n\to\infty} \frac{n^x n!}{x(x+1)\dots(x+n)}$$

$\Gamma(x)$ converges therefore $\lim_{n\to\infty} \frac{n^x n!}{x(x+1)\dots(x+n)}$ also converges, and we have another drastically different form of the Gamma function that is equivalent.

Now if there is a function that may be the Gamma function all we need to do is check the three relations and if it satisfies them it is equal, but if it fails any then it cannot be the Gamma function. In a sense, the Gamma function is defined by the three characteristics, rather than obeying them.

# 4 One Last Definition

Starting with $\Gamma_r(x) = \frac{r^x r!}{x(x+1)\dots(x+r)}$, something that converges to $\Gamma$ when $r \to \infty$

$$\frac{r^x r!}{x(x+1)\dots(x+r)} = \frac{r^x}{x(\frac{x}{1}+1)(\frac{x}{2}+1)\dots(\frac{x}{r}+1)}$$

$$= \frac{e^{xln\ r}}{x(\frac{x}{1}+1)(\frac{x}{2}+1)\dots(\frac{x}{r}+1)} = \frac{e^{x(ln\ r-1-\frac{1}{2}-\frac{1}{3}-\dots-\frac{1}{r})}e^{(x+\frac{x}{2}+\frac{x}{3}+\dots+\frac{x}{r})}}{x(\frac{x}{1}+1)(\frac{x}{2}+1)\dots(\frac{x}{r}+1)}$$

$$= e^{x(ln\ r-1-\frac{1}{2}-\frac{1}{3}-\dots-\frac{1}{r})}.\frac{1}{x}\frac{e^x}{(\frac{x}{1}+1)}\frac{e^{\frac{x}{2}}}{(\frac{x}{2}+1)}\frac{e^{\frac{x}{3}}}{(\frac{x}{3}+1)}\dots\frac{e^{\frac{x}{r}}}{(\frac{x}{r}+1)}$$

$$= \frac{e^{x(log\ r-1-\frac{1}{2}-\frac{1}{3}-\dots-\frac{1}{r})}}{x}.\prod_{i=1}^{r}\frac{e^{\frac{x}{i}}}{(\frac{x}{i}+1)}$$

$\lim_{r\to\infty}(log\ r - 1 - \frac{1}{2} - \frac{1}{3} - \dots - \frac{1}{r}) = -\gamma$ where $\gamma$ is Euler's constant. Despite both being called Gamma, $\gamma$ is unrelated to $\Gamma(x)$ in definition. $\gamma = 0.577215665\dots$ Finally $r \to \infty$

$$\Gamma(x) = e^{-\gamma x}\frac{1}{x}\prod_{i=1}^{\infty}\frac{e^{\frac{x}{i}}}{(\frac{x}{i}+1)} = \lim_{n\to\infty}\frac{n^x n!}{x(x+1)\dots(x+n)} = \int_0^{\infty} e^{-t}t^{x-1}dt$$

[1]

# 5 Differentiation

Using the previously derived equation for $\Gamma$, consider $g(x) = \log\Gamma(x)$

$$g(x) = -\gamma x - log(x) + \sum_{i=1}^{\infty}\left[\frac{x}{i} - log(1+\frac{x}{i})\right]$$

assuming $x > 0$

$$g'(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma - \frac{1}{x} + \sum_{i=1}^{\infty}(\frac{1}{i} - \frac{1}{x+i}) = -\gamma - \frac{1}{x} + \sum_{i=1}^{\infty}\frac{x}{i(x+i)}$$

$g'(x)$ is called the Digamma function $g'(x) = \Psi(x)$. This is not to be confused with $\Psi(a,b,c)$, the function which had been defined for convexity. $\Gamma'(x) = \Psi(x)\Gamma(x)$

For $\Gamma$ to be differentiable we must show that $\Psi(x)$ converges uniformly.

*Proof.* Assume $x > 0$ for now

$$\Psi(x) = -\gamma - \frac{1}{x} + \sum_{i=1}^{\infty}\frac{x}{i(x+i)} = -\gamma - \frac{1}{x} + x\sum_{i=1}^{\infty}\frac{1}{i(x+i)}$$

$$\leqslant -\gamma - \frac{1}{x} + x\sum_{i=1}^{\infty}\frac{1}{i^2} \leqslant -\gamma - \frac{1}{x} + 2x$$

Set $x \leqslant u$, $u$ can be extended as far as needed. Now

$$-\gamma - \frac{1}{x} + 2x \leqslant -\gamma + 2u$$

The upper bound is independent of x and $\Psi$ converges uniformly for $x > 0$. Consider again the equation $\Gamma(x+1) = x\Gamma(x)$. If we differentiate both sides we get

$$\Gamma'(x+1) = x\Gamma'(x) + \Gamma(x)$$

Then dividing by $\Gamma(x+1)$, which never equals zero (easily noticed from exponential form) and using the equation again

$$\frac{\Gamma'(x+1)}{\Gamma(x+1)} = \frac{\Gamma'(x)}{\Gamma(x)} + \frac{1}{x}$$

This relation is valid for all $x$ where $\Gamma(x)$ is defined. Using it in a similar way to when $\Gamma$ is first defined gives $\Psi(x)$ for all negative $x$ where it's defined. [2] □

As $\Psi(x)$ exists, $\Gamma'(x) = \Gamma(x)\Psi(x)$ exists.

Differentiating again gives

$$\frac{d}{dx}\left(\frac{\Gamma'(x)}{\Gamma(x)}\right) = \frac{1}{x^2} + \sum_{i=1}^{\infty}\frac{1}{(x+i)^2} = \sum_{i=0}^{\infty}\frac{1}{(x+i)^2}$$

In fact for $k \geqslant 2$

$$\frac{d^{k-1}}{dx^{k-1}}\left(\frac{\Gamma'(x)}{\Gamma(x)}\right) = \sum_{i=0}^{\infty}\frac{(-1)^k(k-1)!}{(x+1)^k}$$

[2] uses this line of thought but the equation it gives has a mistake in it.

It follows that $\Gamma(x)$ is differentiable arbitrarily many times. Starting with $\Gamma'(x) = \Psi(x)\Gamma(x)$ and differentiating to get more terms, but all of them will be of the form $\Gamma^{(j)}(x)\Psi^{(k)}(x)$. We have $\Psi^{(k)}(x)$ and $j$ is going to be less than that of $\Gamma^{(n)}(x)$ which we are trying to formulate.

## 6 A Bit Of Beta

Another function defined by an integral is the Beta function

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$$

We start with

$$B(x+1,y) = \int_0^1 \left(\frac{t}{1-t}\right)^x (1-t)^{x+y-1}dt$$

then integrating by parts on a similar expression with adjustable interval of integration. $\delta > 0$ and $\epsilon > 0$

$$\int_\delta^{1-\epsilon} \left(\frac{t}{1-t}\right)^x (1-t)^{x+y-1}dt$$

$$= \left[-\frac{(1-t)^{x+y}}{x+y}\left(\frac{t}{1-t}\right)^x\right]_\delta^{1-\epsilon} + \int_\delta^{1-\epsilon}\frac{x}{(1-t)^2}\left(\frac{t}{1-t}\right)^{x-1}\frac{(1-t)^{x+y}}{x+y}dt$$

$$= \frac{-\epsilon^y(1-\epsilon)^x + \delta^x(1-\delta)^y}{x+y} + \frac{x}{x+y}\int_\delta^{1-\epsilon} t^{x-1}(1-t)^{y-1}dt$$

Now $\epsilon \to 0$ and $\delta \to 0$ makes

$$B(x+1,y) = \frac{x}{x+y}B(x,y)$$

This is similar to $f(x+1) = xf(x)$, the first condition for Bohr Mollerup. In fact we can make this true by setting

$$f(x) = B(x,y)\Gamma(x+y)$$

Beta is log convex with respect to $x$ (and $y$). This can easily be seen from the similarity between $\Gamma$ and $B$ in terms of their integral expression. Looking back at the proof $\Gamma(x)$ is convex, since $y$ is a constant, $(1-t)^y$, or the early form of it, is also constant. We assumed $e^{-t} > 0$ and since it does not appear in $B(x, y)$ we will instead use $1(> 0)$.

Furthermore $x$ and $y$ can be permuted with affecting the value of $B(x, y)$. In other words $B(x, y) = B(y, x)$. This is because a substitution of $t = T - 1$ yields $B(x, y) = \int_0^1 T^{y-1}(1-T)^{x-1}dT = B(y, x)$. Therefore, $B(x, y)$ with respect to $y$ is log convex.

$f(x) = B(x, y)\Gamma(x+y)$ is log convex (with respect to $x$) since $B(x, y)$ and $\Gamma(x+y)$ are log convex and the product of any two log convex functions is log convex. We now have the first two conditions for proving that $f(x)$ is the Gamma function. Unfortunately the last property does not hold $B(1, y) = \int_0^1 (1-t)^{y-1}dt = \frac{1}{y}$. Giving

$$f(1) = \frac{1}{y}\Gamma(y+1) = \Gamma(y)$$

Since, again, $\Gamma$ is always positive, we can divide by $\Gamma(y)$ (which is a constant, to make a function that does satisfy all three conditions of Bohr-Mollerup. Hence,

$$F(x) = \frac{B(x, y)\Gamma(x+y)}{\Gamma(y)} = \Gamma(x)$$

Finally

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

An amazing relation between functions. This also further proves that $B(x, y) = B(y, x)$. [2]

Because of the above equality it makes it very easy to calculate $\Gamma(\frac{1}{2})$. Put $x = y = \frac{1}{2}$

$$B\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)} = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{1} = \int_0^1 \frac{1}{\sqrt{t}}\frac{1}{\sqrt{1-t}}dt$$

Now substituting $t = sin^2(x)$ and

$$\int_0^{\frac{\pi}{2}} \frac{1}{sin(x)cos(x)} \cdot 2sin(x)cos(x)dx = \int_0^{\frac{\pi}{2}} 2\ dx = \pi$$

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \pi \quad \Rightarrow \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

[2]

# 7 Stirling's $\sqrt{2\pi}$

Since Gamma has the same growth as $x!$ when $x$ becomes large. It becomes very difficult to calculate $\Gamma(x)$. This is where Stirling's formula comes in useful. It is an approximation for $x!$ for large $x$. It is very easy to establish

$$n^n e^{-n+1} \leqslant n! \leqslant n^{n+1} e^{-n+1}$$

so it would be reasonable to make an approximation that was somewhere in between these boundaries. So we take the geometric mean.

$$\sqrt{n^n e^{-n+1} n^{n+1} e^{-n+1}} = \sqrt{n^{2n+1} e^{-2n-2}} = e \cdot n^{n+\frac{1}{2}} e^{-n}$$

The extra e can be replaced by a constant to make it more accurate. To find this constant we must find $\lim_{n\to\infty} \frac{n!}{n^{n+\frac{1}{2}} e^{-n}}$. To do this we must first prove Wallis formula

**Theorem 11.**

$$\prod_{n=1}^{\infty} \frac{4n^2}{4n^2-1} = \frac{\pi}{2}$$

*Proof.* For $0 < x < \frac{\pi}{2}$,

$$sin^{2n+1}(x) < sin^{2n}(x) < sin^{2n-1}(x)$$

and because $sin^a(x) > 0$ for $a \in \mathbb{N} \setminus \{0\}$

$$\int_0^{\frac{\pi}{2}} sin^{2n+1}(x) < \int_0^{\frac{\pi}{2}} sin^{2n}(x) < \int_0^{\frac{\pi}{2}} sin^{2n-1}(x)$$

Now using the functions and integrals we have already defined, along with repeated use of $f(x+1) = xf(x)$, it can be proven [3] that

$$\int_0^{\frac{\pi}{2}} sin^{2n-1}(x) = \frac{1}{2} B\left(n, \frac{1}{2}\right) = \left(\frac{1}{2}\right) \frac{\Gamma(n)\Gamma(\frac{1}{2})}{\Gamma(n+\frac{1}{2})} = \frac{2 \cdot 4 \cdots (2n-2)}{1 \cdot 3 \cdots (2n-1)}$$

$$\int_0^{\frac{\pi}{2}} sin^{2n}(x) = \frac{1}{2} B\left(n+\frac{1}{2}, \frac{1}{2}\right) = \left(\frac{1}{2}\right) \frac{\Gamma(n+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(n+1)} = \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} \frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} sin^{2n+1}(x) = \frac{1}{2} B\left(n+1, \frac{1}{2}\right) = \frac{1}{2} \frac{\Gamma(n+1)\Gamma(\frac{1}{2})}{\Gamma(n+\frac{3}{2})} = \frac{2 \cdot 4 \cdots (2n)}{1 \cdot 3 \cdots (2n+1)}$$

Now we have

$$\frac{2 \cdot 4 \cdots (2n-2)}{1 \cdot 3 \cdots (2n-1)} < \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} \frac{\pi}{2} < \frac{2 \cdot 4 \cdots (2n)}{1 \cdot 3 \cdots (2n+1)}$$

and

$$1 < \frac{1^2 \cdot 3^2 \cdots (2n-1)^2 (2n+1)}{2^2 \cdot 4^2 \cdots (2n)^2} \frac{\pi}{2} < \frac{2n+1}{2n}$$

now as $n \to \infty$ the right inequality becomes more strict and leaves only the possibility that

$$\prod_{n=1}^{\infty} \frac{4n^2-1}{4n^2}\frac{\pi}{2} = 1$$

which implies

$$\prod_{n=1}^{\infty} \frac{4n^2}{4n^2-1} = \frac{\pi}{2}$$

[3]

□

Make $P_n = \frac{n!}{n^{n+\frac{1}{2}}e^{-n}}$ then

$$\frac{P_n^2}{P_2n} = \frac{(n!)^2 2^{2n}}{(2n)!}\frac{\sqrt{2n}}{n}$$

writing the terms of the denominator and the terms of the numerator give a clearer impression of what is happening as $n \to \infty$.

$$\frac{2 \cdot 4 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)}\frac{\sqrt{2n}}{n}$$

Now expressing Wallis formula as

$$\lim_{n\to\infty} \left(\frac{2 \cdot 4 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)}\right)^2 \frac{1}{2n+1} = \frac{\pi}{2}$$

then

$$\lim_{n\to\infty} \frac{2 \cdot 4 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)}\frac{2}{\sqrt{2n+1}} = \sqrt{2\pi}$$

therefore

$$\lim_{n\to\infty} \frac{P_n^2}{P_2n} = \sqrt{2\pi}$$

As $n \to \infty$, $P_2n = P_n$ meaning

$$\lim_{n\to\infty} \frac{P_n^2}{P_2n} = \lim_{n\to\infty} P_n = \sqrt{2\pi}$$

Now

$$\lim_{n\to\infty} \frac{n!}{n^{n+\frac{1}{2}}e^{-n}\sqrt{2n}} = 1$$

and we have proved Stirling's formula $n! \cong n^{n+\frac{1}{2}}e^{-n}\sqrt{2n}$. Now we can work out n! for large numbers and hence $\Gamma(x+1)$ [3]

This is a very good approximation

$$-\frac{1}{12x}x^{x+\frac{1}{2}}e^{-x}\sqrt{2x} < \Gamma(x-1) < \frac{1}{12x}x^{x+\frac{1}{2}}e^{-x}\sqrt{2x}$$

[3]
Meaning when $x = 10$ Stirling's formula is within a percent of the actual value of $\Gamma(x)$. This approximation gets even better for larger numbers and is within 0.17% when $x = 50$.

## 8 The Zeta Link

The Zeta function, thanks to the similarly named hypothesis, is a very well know and interesting function. It has deep links with the primes and if mastered has major consequences for prime numbers, and hence computer security all over the world. The Zeta function is defined as

$$\zeta(z) = \sum_{r=1}^{\infty} \frac{1}{r^z}$$

It is defined for the complex numbers as well as the reals. It is undefined for $z = 1$ as this is the harmonic series and obviously diverges. It converges for all complex numbers whose real part is greater than one. It is with complex variables though which make this function so interesting. For $Re(\zeta(z)) = 0.5$ is a line in the complex plane for which the conjecture of the Riemann hypothesis is based. For now back to $\Gamma$

Using the integral definition

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt$$

we can substitute $t = ru$

$$\int_0^{\infty} e^{-t} t^{x-1} dt = \int_0^{\infty} e^{-ru} (ru)^{x-1} r \, du = r^x \int_0^{\infty} e^{-ru} u^{x-1} du$$

Next

$$\frac{1}{r^x} = \frac{1}{\Gamma(x)} \int_0^{\infty} e^{-ru} (ru)^{x-1} du$$

Now

$$\zeta(x) = \sum_{r=1}^{\infty} \frac{1}{r^x} = \frac{1}{\Gamma(x)} \sum_{r=1}^{\infty} \int_0^{\infty} e^{-ru} u^{x-1} du = \frac{1}{\Gamma(x)} \int_0^{\infty} u^{x-1} \sum_{r=1}^{\infty} e^{-ru} du$$

$$= \frac{1}{\Gamma(x)} \sum_{r=1}^{\infty} \int_0^{\infty} e^{-ru} u^{x-1} du = \frac{1}{\Gamma(x)} \int_0^{\infty} u^{x-1} \frac{e^{-}u}{1 - e^{-u}} du$$

So finally

$$\zeta(x)\Gamma(x) = \int_0^{\infty} \frac{u^{x-1}}{e^u - 1} du$$

This amazing result means that $\zeta$ and $\Gamma$ are related for all the reals for which they are defined, so not the negative integers, zero or one.
$\Gamma$ actually exists for all complex numbers and is only undefined on the parts

of the real line previously mentioned. In fact the above relationship is taken further and

$$\zeta(z) = \frac{\Gamma(1-z)}{2\pi i} \oint_{u^-} \frac{u^{z-1}}{e^{-u}-1} du$$

for some contour $u^-$, defined everywhere except $z = 1$. [1]

This is unfortunately where Γ's usefulness with the zeta function ends. The study of zeta can go on factoring it out to make things simpler.

We started with $n!$ and we have made it all the way to the forefront of mathematics through the Gamma function. It's usefulness does not end here. I have not mentioned extensively how it helps with similar or seemingly unsimilar integrals. The applications extend through probability, geometry [3] and engineering, but the study of Gamma is not about it's aplications, it's about the steps taken to learn the nature of Gamma beyond a formula.

# References

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