Sampling in Harmonic Analysis

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Sampling, in this context, is the process of digitizing a *continuous signal* by recording samples of its amplitude at regular discrete time intervals, essentially converting the signal into a numeric sequence. Mathematically speaking, sampling is an operator which acts upon an analog signal to produce a digital one [1, p. 3]. By doing this it becomes possible, for example, to store sound recordings on a digital medium, or to simply transfer analogue information where it would not normally be feasible.

The Shannon-Nyquist theorem, which I will later discuss in more detail, states that if we know the highest frequency a signal will reach is β, then the lower bound on the sampling frequency (while still allowing for perfect reconstruction) is 2β. This lower bound is known as the **Nyquist rate**. Similarly, if we know the sampling rate, the theorem gives us the upper bound for the highest frequency of any components of the signal that can be reconstructed perfectly. This upper bound is called the **Nyquist frequency**. [9, *Nyquist rate, Nyquist frequency*]

There are limitations of the theorem, such as bandlimitation of the continuous signal, but these will be discussed later. I will then go on to discuss whether or not these assumptions can ever be satisfied in practice, and the effects this has on the theorem.

1 Fourier Series

To define the notion of a Fourier Series, critical to this section, we first need some fundamental results.

Definition 1. *A function $f : \mathbb{R} \to \mathbb{C}$ is* ***periodic*** *if*

$$x \in \mathbb{R} \Rightarrow f(x + L) = f(x) \text{ for some L} > 0$$

[3, p. 5]

Any such $L > 0$ is called the **period** of f. For example, e^{ix} is periodic, with period 2π, 4π, ...

Lemma 1. *Let $f : \mathbb{R} \to \mathbb{C}$ be periodic with period $L > 0$. Then $F : \mathbb{R} \to \mathbb{C}$, $F(x) := f(Lx)$ is periodic with period 1.*

Proof. This follows from the definition:

$$F(1) = f(L) = f(0 + L) = f(0) = F(0)$$

$\square$

Similarly, if $f(x)$ has period 2π then $F(x) := f(\frac{2\pi}{L}x)$ has period L. We will use this example later when deriving the Fourier series.

Definition 2. *A* ***simple wave*** *is a function ω of the form*

$$\omega_1(x) = c\sin(2\pi kx) \text{ or } \omega_2(x) = c\cos(2\pi kx)$$

where $k \in \mathbb{Z}, c \in \mathbb{C}$. [3, p. 5]

Note that Simple Waves, by definition, have period $1/k$. Also note that c can be (and often is) imaginary. Hence we can write $e^{2\pi ikx}$ as the sum of 2 simple waves using the equality $\cos(2\pi kx) + i\sin(2\pi kx) = e^{2\pi ikx}$.

1.1 Composition of Waves

If you consider, for instance, a sound wave which you hear, it is highly unlikely that this wave is a perfect sinusoidal wave. We can, however, consider this sound wave to be the sum of many (possibly infinitely many) simple sinusoidal waves.

The standard way to view a wave is in the time-domain. Here, the amplitude of the wave is plotted as a function of time. However, we can instead view a periodic function in frequency domain. This involves breaking down the function into its periodic components, and plotting the amplitude of each against its frequency.

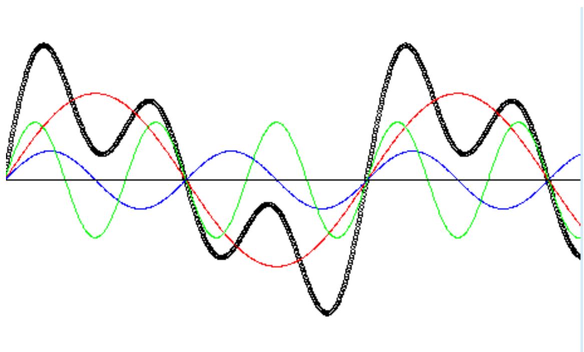

Figure 1: Function (in black) is the sum of the simpler sinusoidal waves

The function above is composed of 3 waves, the red at 1 Hz, blue at 2 Hz, and green at 3 Hz, with amplitudes 30, 10, and 20 respectively. The black wave is the sum of these. Viewing the black wave alone does not tell us much about the components of the function, so instead we look at the frequency domain, or **spectrum** of the function. [11]

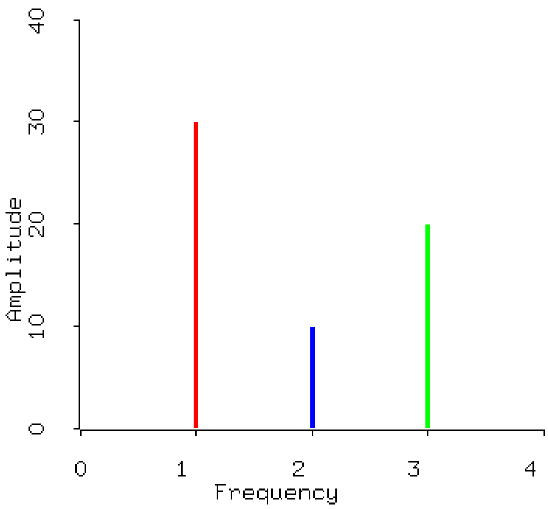

Figure 2: Spectrum of black function above

This spectrum can be plotted if we know the constituent simple waves which make up the function. I discuss other techniques for plotting the spectrum later.

Definition 3. *A **continuous signal** is a function $f : X \to \mathbb{R}; X \subseteq \mathbb{R}$.*

For instance $f(t) := \sin(t)$ is a continuous signal. As in the previous example, we may model a sound wave as a continuous signal.

Definition 4. *A **bandlimited** signal is one where no constituent elements of the signal are periodic with period higher than some frequency W.*

Since the signal may be a sum of many constituent signals, we limit their frequency to some upper limit.

1.2 The Fourier Series

Definition 5. *A **Fourier series** for a function f is a sum of simple waves which converge to f [6]*

The study of Fourier series can be useful in "breaking-up" a more complex periodic function into a weighted sum of simple waves which can be analysed individually, before being recombined to obtain solutions (or approximate solutions) to a the original problem. The coefficients of this weighted sum, denoted c_n, are uniquely determined by the original function, as we will shortly see.

1.3 Deriving the Fourier series of a function

To derive the Fourier series of a function, we calculate a_0, the initial coefficient, then the subsequent values of a_n and b_n for $n \geq 1$.

The following theorem applies to a function f with period 2π on $\mathbb{R}$. This can be generalised by scaling any periodic function, as will be shown later.

In the following proof, I will assume the following standard trigonometric results.

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx = \pi\delta_{mn} \tag{1}$$

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx = \pi\delta_{mn} \tag{2}$$

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)dx = 0 \tag{3}$$

Where the **Kronecker delta** δ_{mn} is an impulse function defined by:

$$\delta_{mn} := \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

Theorem 1 (Canonical Form)**.** *Let $f : [-\pi, \pi] \to \mathbb{R}$ be a continuous function, and suppose that the series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx)) \tag{4}$$

converges uniformly to f for all $x \in [-\pi, \pi]$. Then

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt$$

[6]

Proof. Define the sequence of partial sums

$$s_k(x) := \frac{a_0}{2} + \sum_{m=1}^{k}(a_m \cos(mx) + b_m \sin(mx))$$

of (4) above. Since we assumed s_k converges uniformly to f, we can deduce that, for fixed n, $s_k(x)\sin(nx)$ converges to $f(x)\sin(nx)$ as $k \to \infty$.

Therefore, for fixed n,

$$f(x)\sin(nx) = \frac{a_0}{2}\sin(nx) + \sum_{m=1}^{\infty}(a_m \cos(mx)\sin(nx) + b_m \sin(mx)\sin(nx))$$

Since each term in the integrand is bounded, we can exchange the sum and integration. Integrating term by term using the identities given above, we can see that on the right hand side, the a_m term is always zero, and the b_m term is zero, except for where $m = n$. Hence we have:

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\sin(nx)dx &= \int_{-\pi}^{\pi}\left(\frac{a_0}{2}\sin(nx) + \sum_{m=1}^{\infty}(a_m \cos(mx)\sin(nx) + b_m \sin(mx)\sin(nx))\right)dx \\ &= 0 + \pi b_n\end{aligned}$$

Similarly, multiplying $f(x)$ by $\cos(x)$ in the initial step gives us

$$\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \pi a_n$$

$\square$

Example - The Square Wave
As an example, I will compute the Fourier coefficients of

$$f(x) := \begin{cases} 1 & \text{for } 0 \leq x < \pi \\ -1 & \text{for } \pi \leq x \leq 2\pi \end{cases}$$

Since $f(x)$ is odd, then $f(x)\cos(x)$ is also odd, so $a_n = 0 \; \forall n \in \mathbb{N}$. For $n \geq 1$, b_n is given by

$$b_n = \frac{1}{\pi}\left((1)\int_0^{\pi} \sin(nx)dx + (-1)\int_{\pi}^{2\pi} \sin(nx)dx\right) = \begin{cases} \frac{4}{n\pi} & \text{for odd n} \\ 0 & \text{for even n} \end{cases}$$

Hence we have

$$f(x) = \frac{4}{\pi}\left(\frac{\sin(x)}{1} + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + ...\right)$$

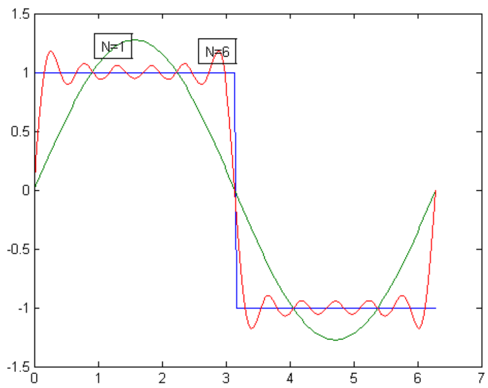

Figure 3: A plot of the Fourier series of the square wave function, including just the first term, then just the first 6 terms.

1.4 Exponential Form

Lemma 2. *By using trigonometric identities, it is equivalent to write*

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$$

where c_n is given by

$$
\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dx = \frac{a_n}{2} - \frac{b_n i}{2} \\
c_0 &= \frac{a_o}{2} \\
\textit{and } c_{-n} &= \frac{a_n}{2} + \frac{b_n i}{2}
\end{aligned}
$$

Proof. We know

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}, \sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

So,

$$
\begin{aligned}
f(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \frac{e^{int} + e^{-int}}{2} + b_n \frac{e^{int} - e^{-int}}{2i} \right) \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{a_n - ib_n}{2} e^{int} + \frac{a_n + ib_n}{2} e^{-int} \right)
\end{aligned}
$$

Writing

$$
\begin{aligned}
c_0 &= \frac{a_0}{2} \\
c_n &= \frac{a_n - ib_n}{2} \\
c_{-n} &= \frac{a_n + ib_n}{2}
\end{aligned}
$$

gives

$$
\begin{aligned}
f(t) &= c_0 + \sum_{n=1}^{\infty} \left(c_n e^{int} + c_{-n} e^{-int} \right) \\
&= \sum_{n=0}^{\infty} c_n e^{int} + \sum_{n=-\infty}^{-1} c_n e^{int} \\
&= \sum_{n=-\infty}^{\infty} c_n e^{int}
\end{aligned}
$$

□

1.5 Functions of Arbitrary Period

For a function periodic on $[-L, L]$ rather than $[-\pi, \pi]$, we can use a change of variables: $x = \frac{\pi x'}{L}$ so $dx = \frac{\pi dx'}{L}$.

Rearranging gives $x' = \frac{Lx}{\pi}$ and plugging this in gives

$$f(x') = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x'}{L}\right) + b_n \sin\left(\frac{n\pi x'}{L}\right)\right)$$

Therefore

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x')dx' \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x')\cos\left(\frac{n\pi x'}{L}\right)dx' \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x')\sin\left(\frac{n\pi x'}{L}\right)dx' \end{aligned}$$

In a similar way, exponential form becomes:

$$\begin{aligned} f(x') &= \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x'}{L}} \\ \text{where } c_n &= \frac{1}{2L}\int_{-L}^{L} f(x') e^{\frac{-in\pi x'}{L}} dx' \end{aligned}$$

For the integration limits, any region of width $2L$ can be used. [13]

1.6 Convergence of Fourier Series

If f is any integrable function, we may compute the Fourier coefficients a_n and b_n (or just c_n), however in general there is no guarantee that the sum will converge uniformly to the original function f. The ability of the Fourier series to represent a particular function $y = f(t)$ is given by the Dirichlet conditions, named after the German mathematician Gustav Dirichlet. The Dirichlet conditions are sufficient conditions to ensure existence and convergence of a Fourier series. [2, p. 20] [10]

- $\int_{-L}^{L} |f(t)|dt < \infty$
- In one period, $f(t)$ has only a finite number of minima and maxima.
- In one period, $f(t)$ has only a finite number of discontinuities and each one is finite.

If all 3 conditions are satisfied, the Fourier expansion of f converges to f at all points where f is continuous, and to $\frac{1}{2}(f(t_n+) + f(t_n-))$ at points t_n where f is discontinuous.

In theory, signals that violate these conditions can be constructed. However, it is not possible to create such a signal that violates these conditions in a lab. Therefore we can assume that in sampling a signal, our signal will have a valid Fourier series representation.

2 Fourier Transform

2.1 Fourier Series to Fourier Transforms

Fourier series deal only with periodic functions, but non-periodic functions can also be broken down into Fourier components in a process known as the Fourier Transform. We do this by considering a function periodic on $[-L, L]$ and letting $L \to \infty$.[2, p. 111]

Suppose

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}} \\ &\text{and} \\ c_n &= \frac{1}{2L}\int_{-L}^{L} f(x)e^{\frac{-in\pi x}{L}}dx \\ \text{now, set } \omega_n &= \frac{n\pi}{L} \\ \text{and } \Delta\omega &= \omega_n - \omega_{n-1} = \frac{\pi}{L} \end{aligned}$$

Substituting in the integral formula of the c_n, we get

$$f(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left(e^{i\omega_n x}\int_{-L}^{L} e^{-i\omega_n t}f(t)dt\right)\Delta\omega$$

Now, taking the limit as $T \to \infty$ (and hence $\Delta\omega \to 0$), the sum becomes an integral, the discrete ω_n become a continuous variable ω and we get

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(e^{i\omega x}\int_{-L}^{L} e^{-i\omega t}f(t)dt\right)d\omega$$

We can then split this up to obtain the Fourier transform pair

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}d\omega \\ \text{and } F(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \end{aligned}$$

The Fourier transform, $F(\omega)$ gives us the *spectrum* of f.

2.2 Dirac Delta

Consider

$$f(t) := \begin{cases} \frac{1}{k} & \text{for } 0 \le t \le k \\ 0 & \text{for } t > k \end{cases}$$

The area enclosed by this graph is given by $k\frac{1}{k} = 1$ for any value of k. Now let $k \to \infty$. We obtain a vertical line or "impulse" at $t = 0$ with the height of the line tending to infinity. This is said to be a "unit impulse" since the area enclosed by it is still 1. [2, p. 116]

This function is called $\delta(t)$, the Dirac delta function. The Fourier Transform of $\delta(t)$, $\mathscr{F}[\delta(t)]$ is given by

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} \delta(t) e^{-i\omega t} dt \\ &= \int_{0-}^{0+} \delta(t) e^{-i\omega t} dt \\ &= \int_{0-}^{0+} \delta(t).1 \, dt \\ &= 1 \end{aligned}$$

So for all frequencies ω, $F(\omega) = 1$. Hence the impulse function $\delta(t)$ contains all the possible frequencies as its components.

2.3 Calculating Fourier Transforms

Consider the constant function $f(t) = A$ for some $A \in \mathbb{R}$. The area under this graph does not converge, so attempting to calculate the Fourier transform will result in an indeterminate quantity:

$$\begin{aligned} F(\omega) = \mathscr{F}(A) &= \int_{-\infty}^{\infty} A e^{i\omega t} dt \\ &= A \left[\frac{e^{-i\omega t}}{-i\omega} \right]_{t=-\infty}^{t=\infty} \end{aligned}$$

This expression is indeterminate. This problem also occurs when trying to take the Fourier transform of a sinusoidal function.

To overcome this, and determine the Fourier transform of the constant function, consider instead $f(t) = Ae^{-at}$ for $t > 0$ and $f(t) = Ae^{at}$ for $t < 0$, where $a > 0$. This has finite enclosed area. I shall calculate the Fourier transform of this, then take the limit as $a \to 0$ to obtain the transform of the constant function.

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{0} A e^{at} e^{-i\omega t} dt + \int_{0}^{\infty} A e^{-at} e^{-i\omega t} dt \\ &= A \left[\frac{e^{t(a-i\omega)}}{(a-i\omega)} \right]_{-\infty}^{0} + A \left[\frac{e^{-t(a+i\omega)}}{-(a+i\omega)} \right]_{0}^{\infty} \\ &= \frac{A}{a-i\omega} + \frac{A}{a+i\omega} \\ &= \frac{2Aa}{a^2+\omega^2} \end{aligned}$$

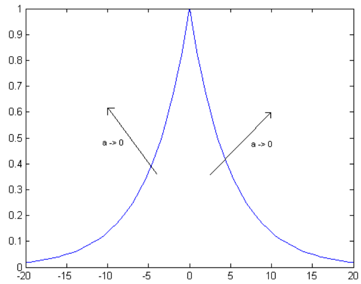

Figure 4: A plot of $y = f(t)$ defined as above

Consider the case where $\omega = 0$ to see

$$= \frac{2A}{a + \frac{0}{a}} \longrightarrow \infty \text{ as } a \longrightarrow 0$$

So we see that here too the Fourier transform is an impulse function. We can determine the strength of the impulse by integrating $F(\omega)$ over all possible frequencies ω

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{2Aa}{a^2+\omega^2} d\omega &= 4Aa \int_{0}^{\infty} \frac{1}{a^2+\omega^2} d\omega \\
&= 4Aa \left[\frac{1}{a} \tan^{-1} \frac{\omega}{a} \right]_0^{\infty} \\
&= 4A \left[\frac{\pi}{2} - 0 \right] \\
&= 2A\pi
\end{aligned}$$

Hence the Fourier transform of a constant function A is an impulse of strength $2A\pi$, i.e.

$$\mathscr{F}(A) = F(\omega) = 2A\pi\delta(\omega)$$

The spectrum of A is a single line, of amplitude $2A\pi$ at $\omega = 0$.

3 The Shannon-Nyquist Theorem

Harry Nyquist (1889-1976) was a Swedish physicist who did extensive research into information theory while working for firstly AT&T labs in the USA from 1917 until 1934, and subsequently at Bell South until his retirement 20 years later. Whilst working in telecommunications, he did research into television, facsimile technology and error correction models, but delved later into information theory. [7]

The foundations of information theory were very much the work of Claude Shannon (1916-2001), also an engineer at AT&T laboratories. He was known by his colleagues as an eccentric genius, but one who would often keep to himself.

He was frequently seen traversing the corridors of AT&T on a unicycle, sometimes also juggling to compound the problem. His obituary in *The Times*[12] comments on his invention of the unicycle with an off-centre hub, causing him to bob up and down as he traversed the corridors, much to the amusement of his colleagues.

The Shannon-Nyquist Theorem is often called just the Shannon Sampling Theorem, named after Shannon, as, although Nyquist hypothesised the result in 1928, it was not until 1949 when the result was proven by Shannon. Nyquist's work in this regard lead to the sampling rate required by the theorem to be called the *Nyquist rate*.

According to the theorem, it is possible to fully reconstruct the signal by interpolating the function between points sampled at this Nyquist rate. It should be intuitively clear that increasing the sample rate will increase the accuracy of the reconstruction, regardless of the interpolation used. For example, in the figures 5 through 7 I have linearly interpolated between points of Sin(x), doubling the sampling frequency each time.

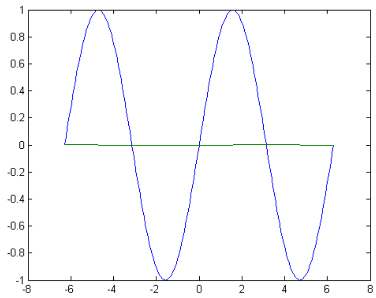

Figure 5: Samples taken every π

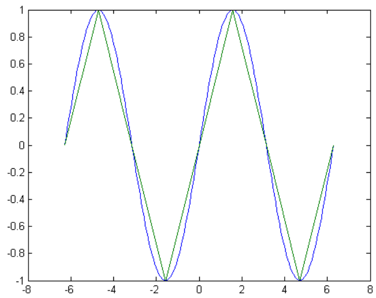

Figure 6: Samples taken every $\pi/2$

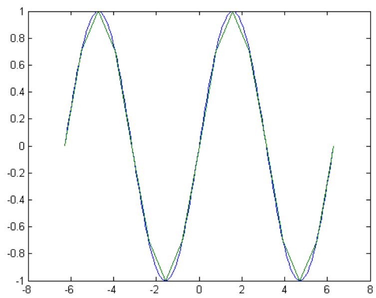

Figure 7: Samples taken every $\pi/4$

The reconstructed function becomes more accurate as the sampling rate increases. The Shannon-Nyquist theorem gives us a lower bound on the sampling rate required to reproduce the function completely.

To reconstruct the signal from the samples, the Whittaker-Shannon interpolation method is used.

Theorem 2 (The Sampling Theorem in the Time Domain)**.** *If a function $f(t)$ contains no frequencies higher than some frequency W, then it is completely determined by giving its samples taken every $\frac{1}{2W}$ [4, p. 67]*

That is to say that the highest frequency signal that can be perfectly reconstructed is one half of the sample rate.

Proof. Shannon's original proof runs as follows:
Let $F(\omega)$ be the frequency spectrum of $f(t)$. Then

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-2\pi W}^{2\pi W} F(\omega)e^{i\omega t}d\omega \end{aligned}$$

since we assume the spectrum $F(\omega)$ to be 0 outside the band $\pm W$ (ie bandlimited).
Now, if we let $t = \frac{n}{2W}$ where $n \in \mathbb{Z}$, we obtain

$$f(\frac{n}{2W}) = \frac{1}{2\pi}\int_{-2\pi W}^{2\pi W} F(\omega)e^{i\omega\frac{n}{2W}}d\omega$$

On the left hand side are the values of f at the samping points $t = \frac{n}{2W}$. Notice that on the right hand side we have the exponential form of the nth term of the Fourier series for $F(\omega)$ where W is the period of F. This means that the sampling points on the left completely determine the Fourier coefficients of $F(\omega)$ on the right, and hence they completely determine F. Since $F(\omega)$ completely determines the original signal $f(t)$, the original samples determine $f(t)$ completely. $\square$

3.1 Aliasing

In the case where a sampling rate lower than the required *Nyquist rate* is used, a phenomenon known as *aliasing* occurs. This can occur whenever a continuous signal is discretised. For example, consider the orbit of the Earth around the Sun. From the Earth we view the Sun move from east to west through the course of the day. However, if we took a sample of the Sun's position in the sky every 23 hours, the Sun would appear to move from west to east in our series of photographs, at $\frac{1}{24}$th speed that it really moves from east to west. This concept underpins aliasing, that is to say that more than one continuous signal can be represented by the same set of samples.

Definition 6. *In signal processing, a signal that is not sampled sufficiently often may suffer from **aliasing** when reconstructed.*

Aliasing causes different continuous signals to be represented in the same way when reconstructed from their samples.

Since the theorem requires the assumption that the signal is bandlimited, any sampled signal must not contain frequency elements above twice its sampled rate, or these elements must be negligibly small and not affect the reconstruction in order to be perfectly reproduced. In practise, it is impossible to guarantee this, so certain techniques are used to minimize their impact on the reproduction. One such technique is filtering the original signal. When processing the signal, a "brick-wall" or "low-pass" filter can remove certain frequencies with varying degrees of accuracy in order to limit their effects on aliasing of the signal that may reduce the accuracy of the reproduction.

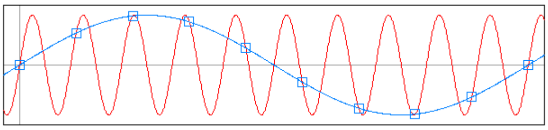

Figure 8: The red function is sampled at a low sampling rate which when reconstructed could give an alias (the blue function) which is not equal to the original

4 Usage of the Theorem

4.1 Audio Recordings

The human voice typically has a frequency of between 500-2000Hz, and we can hear frequencies up to around 20kHz. For CDs, the sample rate was chosen to be 44,100Hz, thus giving a theoretical maximum recordable sound frequency of 22,050Hz which is just outside the audible range for most human beings. This also gives rise to *oversampling*, which is sampling the signal at a rate above the Nyquist rate. This allows for error correction algorithms, such as the Reed−Solomon algorithm, to deduce lost data points should the CD become scratched or otherwise unreadable.

There is, however, another issue with CDs, and that is the accuracy of the samples. CDs are recorded using 16-bit samples. This gives us $2^{16} = 65536$ possible 'amplitudes' that can be recorded. Since the wave is continuous, it can take any value in between these 65536 pre-determined levels, and hence information is lost due to rounding.

For this application, perfect reconstruction is not necessary, so long as the reproduction of the original wave is *near enough* to not be noticed by the listener. The choice of 16-bit samples balances accuracy with space requirements on the disc, since more accurate samples take up more space.

4.2 Telecommunications

Telephone lines were designed to carry voice data, and so are limited to frequencies less than around 4000Hz. According to Nyquist, it would take 8000 samples each second to record this 4000Hz signal perfectly. Conversely, this also means we can send 8000 samples worth of data across the telephone circuit each data. However, due do reliability and noise issues on telephone lines, only 8-bit sampling is used. This makes the gap between sampling levels larger, so slight variations in the signal will not affect the data. 8 bits * 8,000 samples/sec = 64,000 bits per second, which is the theoretical maximum throughput on a standard telephone line. [8]

5 Conclusion

We have seen two explicit examples of where the sampling theorem is used above, but there are many more. In this age of communication, the process of digitizing signals, and undoing the digitization is a task that millions of devices worldwide do on a daily basis. New digitial television signals, introduced recently, are analogue, yet carry digital data in the form of samples. The sampling theorem allows us to predetermine, for instance, how many frequencies to reserve for transmission of a certain signals, depeneding on how much data needs to be sent. It can also be used to calculate the digital data transfer rate over various analogue media.

There are many limitiations to the theorem. For one, the theorem does not really tell us what happens if the prerequisites are not met. An analysis into this would reveal more about aliasing, as mentioned above.

I could go on to discuss reconstructing the signal from the samples, but this method - the "Whittaker-Shannon Interpolation Method" would require substantial discussion.

Whilst the result of the theorem has been suspected for over a century now, Shannon's work on the proof has led to it being regarded as one of the most important results in information theory, where I think it will remain for a long time to come.

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