Second Year Essay:
The Logistic Map, Period Doubling Cascades and the Onset of Chaos

April 22, 2007

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1 Introduction

"The scientist does not study nature because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful. If nature were not beautiful, it would not be worth knowing, and if nature were not worth knowing, life would not be worth living. Of course I do not here speak of that beauty that strikes the senses, the beauty of qualities and appearances; not that I undervalue such beauty, far from it, but it has nothing to do with science; I mean that profounder beauty which comes from the harmonious order of the parts, and which a pure intelligence can grasp."
Henri Poincaré

1.1 Introduction

This essay will investigate an equation known as the Logistic Map, equivalently known as the Verhulst Equation. The Verhulst Equation is a differential equation that was originally derived by Pierre Francois Verhulst in the 1830s to model the growth of a population of a species, and is still widely used today in population dynamics as a more basic model. This equation has been so influential in the past that Lotka branded it the "Law of Population Growth" upon deriving it for himself in the 1920s. The equation is a simple version of the more general (competitive) Lotka-Volterra equations.

As a model, the equation has come under some criticism more recently since, for example, it was once used to predict that the world population could not exceed 2 billion, and then when this number was surpassed it was tweaked to predict that the world population could not go exceed 2.6 billion[1].

However it is not so important to worry about how accurate the model actually is, since the equivalent logistic map has arguably become more famous for a completely different reason, concerned with a branch of Mathematics that is popular today. The logistic map is a quadratic map, however its simplicity is deceiving in as much that, if given the correct conditions, it can be used to exhibit intricate properties such as chaos.

The Verhulst Equation is given by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \lambda P \left(1 - \frac{P}{K}\right)$$

where P is the population as a function of the time t; λ is a positive constant denoting birth rate, determined by various conditions on the population including the rate of reproduction and the amount of food available in the environment; and where K is another positive constant denoting the carrying capacity, the maximum population that can be supported by the environment.

[1] The website `http://www.un.org/News/Press/docs/2005/pop918.doc.htm` shows how these predictions were very much incorrect!

Equivalently, it can be expressed as the difference equation

$$x_{n+1} = \lambda x_n(1 - x_n) \quad \text{where} \quad 0 \leq x_n \leq 1 \text{ and } \lambda > 1$$

known as the logistic map. This will be the form of the equation that shall be used throughout this essay. λ, as with the Verhulst Equation, denotes the birth rate; while x_n denotes the population size P at year n as a proportion of the carrying capacity K (hence x_n is restricted for practical purposes to $0 \leq x_n \leq 1$).

This essay will primarily focus on what happens as λ is varied in the logistic map. It will become evident that for some values of λ, perturbing the value of the initial population x_0 even by a tiny amount will make the long-term behaviour of the solution vastly different, thus making the behaviour seem almost "random". Therefore, applying the theory to population dynamics it will be shown that, for species with these birth rates λ, it would be practically impossible (with current technology at least) to predict what the population of a species would be in a number of years time. This sensitive reliance on the given initial conditions will be proved using the fact that chaos can be exhibited with the logistic map.

1.2 Definitions

Before beginning to investigate the map, it is important to define a few fundamental notions:

Definition 1 (Fixed point). A fixed point x_* of a difference equation $x_{n+1} = f(x_n)$ is given by $f(x_*) = x_*$

Definition 2 (Stability of fixed points). A fixed point x_* of a difference equation $x_{n+1} = f(x_n)$

1. Is stable if $|f'(x_*)| < 1$, and
2. Is unstable if $|f'(x_*)| > 1$

Definition 3 (Periodic point). A point x of a difference equation $x_{n+1} = f(x_n)$ that satisfies the equation $f^n(x) = f(f(...(x))) = x$ (f applied to x n times)[2] for some $n \in \mathbb{N}$ is called a periodic point. The smallest value of n for which $f^n(x) = x$ holds is called the period.

Iterating a fixed point of f by applying f will just give the same point back. However, if iterating a periodic point of f, say x with period q, x will only be returned after f has been applied to x q times, and thus clearly the same finite sequence of q numbers will keep repeating as f is applied to x indefinitely (since the $(q+1)$th iteration of x will just be f applied to $f^q(x)$ which by definition is just x).

[2] A more rigorous definition of $f^n(x)$ is given at the start of the next section

2 Fixed Points of the Logistic Map

In order to make the notation much simpler to understand, the function p will be defined as follows:

$$p : [0,1] \to \mathbb{R} \qquad p(x_n) = \lambda x_n(1 - x_n),$$

so then $x_{n+1} = p(x_n)$. The convenient notation for iterating the function f, as described in section 1.2, shall also be employed throughout this essay. That is, $f^k(x_0)$ will be inductively defined to be

$$f^1(x_0) = f(x_0), f^2(x_0) = f(f(x_0)),\ ...,\ f^k(x_0) = f(f^{k-1}(x_0))$$

.

2.1 Periodic Points of $p(x_n)$, With Period One

2.1.1 Analysis of the Fixed Points of $p(x_n)$

Lemma 1. *p has fixed points at $x_* = 0$ and at $x_* = \frac{\lambda-1}{\lambda}$*

Proof: The fixed points x_* of the logistic map (and indeed any general difference equation) are given by $p(x_*) = x_*$. Hence, to find the fixed points of p, the equation $\lambda x_*(1 - x_*) = x_*$ needs to be solved. This is equivalent to finding the solutions of the equation $\lambda x_*^2 + x_*(1 - \lambda) = 0$. Factorising gives the equation $x_*[\lambda x_* - (\lambda - 1)] = 0$. Therefore the solutions of the equation are $x_* = 0$ and $x_* = \frac{\lambda-1}{\lambda}$. □

This Lemma proves where the fixed points of the logistic map are. Now a stronger result will be proved:

Proposition 2. *The fixed point $x_* = 0$ of p is always unstable, and the fixed point $x_* = \frac{\lambda-1}{\lambda}$ of p is stable only if $1 < \lambda < 3$.*

Proof: $p'(x_n) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\lambda x_n(1 - x_n)\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\lambda x_n - \lambda x_n^2\right)\right) = \lambda(1 - 2x_n)$. So $p'(0) = \lambda > 1$ making $x_* = 0$ a stable fixed point of $p(x_n)$, and $p'(\frac{\lambda-1}{\lambda}) = 2 - \lambda$, making $x_* = \frac{\lambda-1}{\lambda}$ a stable fixed point of $p'(_n)$ only if $1 < \lambda < 3$. □

2.1.2 Interpretation of the Fixed Points of $p(x_n)$

So it has been proved that, whenever $1 < \lambda < 3$, x_n moves towards the fixed point $x_* = \frac{\lambda-1}{\lambda}$ and away from the fixed point $x_* = 0$ as it is iterated by p. This shows that the population, over time, will settle down to the finite number $\frac{\lambda-1}{\lambda}$. This is illustrated in Figure 1 for when $\lambda = 2.8$.

This graph, of x against both $p(x)$ and x, shows what happens to x_n as p is repeatedly applied to some arbitrary starting population x_0. Starting at x_0 on the x-axis in the domain and moving vertically until the graph $y = p(x)$ is met finds the value of $x_1 = p(x_0)$. Moving across horizontally until the line $y = x$ is

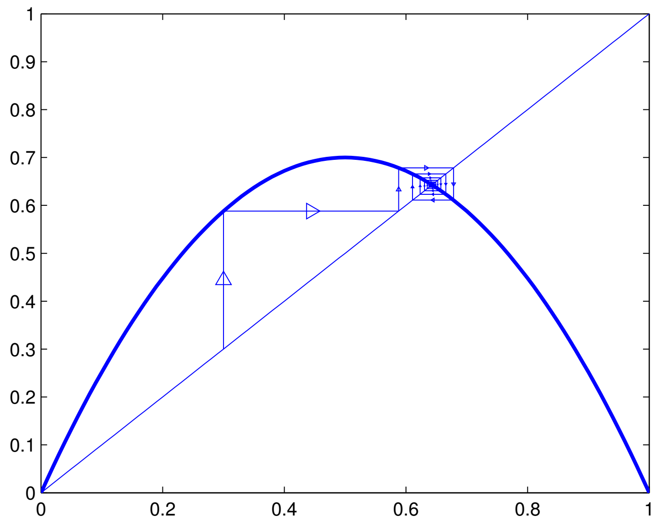

Figure 1: The cobweb diagram for $\lambda = 2.8$ and $x_0 = 0.3$, showing the first 50 iterates of x_0. Note that $\frac{2.8-1}{2.8} \approx 0.643$, which is the approximate y-coordinate of the point that the cobweb diagram converges to. [1]

met puts us at the point x_1 on the x-axis, then moving vertically again until the graph $y = p(x)$ is met finds the value of $x_2 = p(x_1)$. This process is repeated indefinitely to show what happens as $n \to \infty$.

Proposition 2 also gives further justification for restricting λ to being strictly greater than 1 rather than to just being positive. If $0 < \lambda < 1$ then by the results in Proposition 2, $x_* = 0$ would be a stable fixed point of p, and $x_* = \frac{\lambda-1}{\lambda}$ would be an unstable fixed point of p. Hence, x_n would move away from the fixed point $x_* = \frac{\lambda-1}{\lambda}$ and would move towards the fixed point $x_* = 0$ as it is iterated by p. This merely represents a population that would gradually die out (i.e. Settle down to a population of zero). This situation isn't very interesting and needs no investigation.

This leaves one natural question: What happens as λ is increased through the point $\lambda = 3$?

When $\lambda = 3$, $p'(\frac{\lambda-1}{\lambda}) = 2 - \lambda = -1$. This fixed point will change from being stable to being unstable if it is increased any further. Then there would be two unstable fixed points of p and no stable fixed points of p, presumably making the behaviour of x_n more complicated as n is increased. The next section attempts to understand this more complicated behaviour.

2.2 Periodic Points of $p(x_n)$, With Period Two

Turning attention away from p for a moment, it is immediately evident from drawing a simple graph that there is something special about p^2 at $\lambda = 3$. If $\lambda \leq 3$ then the graphs (restricted to $0 < x < 1$) of $y = p^2(x) = p(p(x))$ and $y = x$ intersect once, but if $\lambda > 3$ then the graphs intersect three times [2]. This is illustrated in Figures 2 and 3.

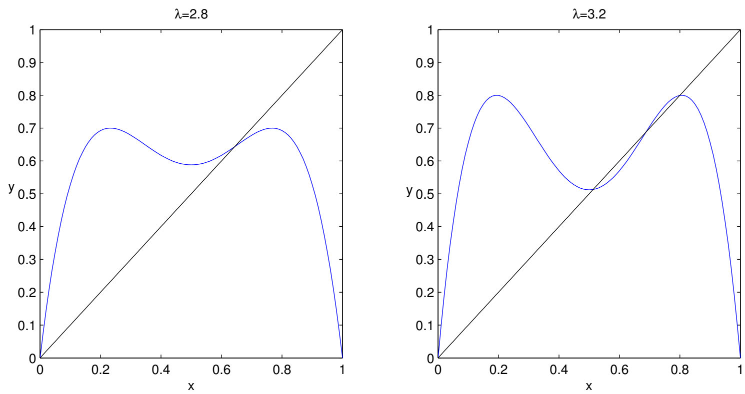

Figure 2: The graphs of p^2 and x against x, for $\lambda = 2.8$ (left) and $\lambda = 3.2$ (right)

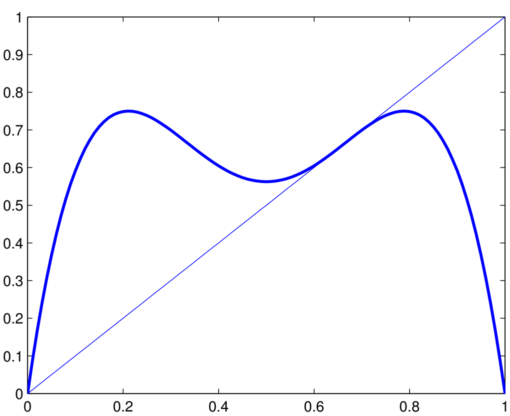

Figure 3: The graphs of p^2 and x against x, for $\lambda = 3$

These graphs show an example of a bifurcation. As λ is increased, new solutions to the equation $p^2(x) = x$ on $0 < x < 1$ are created. In other words, two (new) fixed points of p^2 - periodic points of p with period two - are created. How can this heuristic argument be justified, and proof be obtained to show that new fixed points are really created? Proposition 3 does just that.

2.2.1 Analysis of the Fixed Points of $p^2(x_n)$

Proposition 3. *$p^2(x_n)$ has four fixed points. Two of these fixed points are also fixed points of $p(x_n)$ and the other two new fixed points are distinct and exist in $\mathbb{R}$ only if $\lambda > 3$.*

Proof: Firstly for the formality of calculating $p^2(x_n)$. $p^2(x_n) = p(p(x_n)) = \lambda(\lambda x_n(1 - x_n))(1 - (\lambda x_n(1 - x_n))) = \lambda^2 x_n(1 - x_n)(1 - \lambda x_n + \lambda x_n^2)$. So, by equating this to x_n, it is easy to see that the desired equation in x that finds the fixed points of $p^2(x_n)$ is

$$\lambda^2 x(1 - x)(1 - \lambda x + \lambda x^2) - x = 0 \tag{1}$$

Consider the fixed points of p. If x_* is a fixed point of $p(x_n)$ then $p^2(x_*) = p(p(x_*)) = p(x_*) = x_*$. This proves that if x_* is a fixed point of $f(x)$ then it will also be a fixed point of $f^2(x)$. A simple inductive argument could be applied to show that x_* would also be a fixed point of $f^k(x)$ for any $k \in \mathbb{N}$. This proves that the pair of fixed points of p are also fixed points of p^2. Therefore it remains to show that the two new fixed points are created.

The two known fixed points of p^2 can be used to determine the other two. Two factors of equation (1) will be x and $\left(x - \frac{\lambda-1}{\lambda}\right)$, corresponding to the fixed points at $x = 0$ and $x = \frac{\lambda-1}{\lambda}$ respectively. It can be verified, by multiplying out, that equation (1) is equivalent to

$$-\lambda^3 x\left(x - \frac{\lambda - 1}{\lambda}\right)\left[x^2 - \left(\frac{\lambda + 1}{\lambda}\right)x + \left(\frac{\lambda + 1}{\lambda^2}\right)\right] = 0$$

so the new roots can easily be obtained by setting the quadratic factor of the equation to 0 and solving for x. The new roots are then given by

$$x = \frac{(\lambda + 1) \pm \sqrt{(\lambda + 1)(\lambda - 3)}}{2\lambda}.$$

Observe that the discriminant of the new roots, $(\lambda + 1)(\lambda - 3)$, is positive iff $\lambda > 3$. The new roots are distinct and in $\mathbb{R}$ iff $\lambda > 3$. This completes the proof. □

Now, as with the two fixed points in Lemma 1, a stronger result will be proved:

Proposition 4. *Given $\lambda > 3$, the fixed points $x_* = 0$ and $x_* = \frac{\lambda-1}{\lambda}$ of p^2 are always unstable, and the other two fixed points of p^2 are stable only if $3 < \lambda < (1 + \sqrt{6})$.*

Proof: Firstly $[p^2(x_n)]'$ needs to be calculated. By the chain rule, $[p^2(x_n))]' = p'(p(x_n)).p'(x_n) = \lambda[1-2(\lambda x_n(1-x_n)].\lambda(1-2x_n) = \lambda^2(1-2x_n)(1-2\lambda x_n + 2\lambda x_n^2)$.

Evaluating p^2 at the values of each of the four fixed points:

$x_* = 0$:

$$[p^2(0)]' = \lambda^2 > 9 > 1$$

$x_* = \frac{\lambda-1}{\lambda}$:

$$\begin{aligned}
\left[p^2\left(\frac{\lambda-1}{\lambda}\right)\right]' &= \lambda^2\left(1-\frac{2(\lambda-1)}{\lambda}\right)\left(1-\frac{2\lambda(\lambda-1)}{\lambda}+\frac{2\lambda(\lambda-1)^2}{\lambda^2}\right) \\
&= [\lambda-2(\lambda-1)]\left[\lambda-2\lambda(\lambda-1)+2(\lambda-1)^2\right] \\
&= (2-\lambda)\left(\lambda-2\lambda^2+2\lambda+2\lambda^2-4\lambda\ +2\right) \\
&= (2-\lambda)(2-\lambda) \\
&> 1 \qquad \forall\ \lambda > 3
\end{aligned}$$

Thus both $x_* = 0$ and $x_* = \frac{\lambda-1}{\lambda}$ are always unstable fixed points of p^2.

$x_* = \frac{(\lambda+1)\pm\sqrt{(\lambda+1)(\lambda-3)}}{2\lambda}$:

Because the algebra becomes particularly cumbersome (especially when typesetting in LaTeX) the detailed calculation of $[p^2(x_*)]'$ will be omitted from this proof. Instead, I shall tell you that $[p^2(x_*)]' = -\lambda^2+2\lambda+4$ (regardless of the $\pm$ sign), and append the detailed proof of this result.

To work out when $|[p^2(x_*)]'| < 1$ a little further investigation is required. Consider the function $g : [3,\infty) \to \mathbb{R}$ defined by $g(l) = -l^2+2l+4$. $g'(l) = -2l+2 = 2(1-l)$ which is negative for all l in the domain, meaning that $g(l)$ is always strictly decreasing. Furthermore $g(3) = 1$ meaning that $g(l)$ will take values strictly between ± 1 for all $l \in (3,a)$, where $a > 3$ is a constant that needs to be determined, and nowhere else.

This function g guarantees that $[p^2(x_*)]'$ will take values strictly between ± 1 only for all $\lambda \in (3,a)$ for some $a > 3$. Consequently, g shows that $|[p^2(x_*)]'| < 1$ only for all $\lambda \in (3,a)$, meaning that the new fixed points of p^2 are stable only for all $\lambda \in (3,a)$. Therefore it is left to show that $a = (1+\sqrt{6})$.

To calculate a the solution to the equation $g(a) = -1$ is required, the point at which $[p^2(x_*)]'$ changes from being stable to being unstable. $g(a) = -1 \Leftrightarrow$

$-a^2 + 2a + 4 = -1 \Leftrightarrow a^2 - 2a + 5 = 0$. Therefore $g(a) = -1$ if $a = \frac{2 \pm \sqrt{4+20}}{2} = (1 \pm \sqrt{6})$. Since $(1 - \sqrt{6}) < 3$, the sole valid solution to the equation $g(a) = -1$ is $a = (1 + \sqrt{6})$.

Thus $|[p^2(x_*)]'| < 1$ only $\forall\ \lambda \in (3, (1 + \sqrt{6}))$, completing the proof. □

2.2.2 Interpretation of the Fixed Points of $p^2(x_n)$

At this point recall the definitions in Section 1.2. The important difference between a fixed point and a periodic point with period p was noted. The full long-term behaviour of the function can be exlained using this difference. The stable fixed points of p^2 are *not* fixed points of p, rather these stable fixed points are periodic points of p with period two. Therefore the theory suggests that whenever $3 < \lambda < (1 + \sqrt{6})$ the population x_n will (eventually) alternate between the two values $\frac{(\lambda+1) \pm \sqrt{(\lambda+1)(\lambda-3)}}{2\lambda}$ forever, i.e. alternate between values that will depend solely on the value of λ.

This is illustrated in Figure 4, the same cobweb diagram as before but with $\lambda = 3.2$. As $n \to \infty$, the population x_n will oscillate between two values forever, which is shown by the fact that the line eventually starts to go around on the same rectangular path indefinitely. These two values, $\frac{(\lambda+1) \pm \sqrt{(\lambda+1)(\lambda-3)}}{2\lambda}$, are given by where the two horizontal edges of the 'rectangle' would meet the y-axis supposing they were extended.

2.3 Periodic Points of $p(x_n)$, With Higher Periods

At this point, as when finding the "special" value $\lambda = 3$, a natural question arises: What happens when λ is increased through the point $\lambda = (1 + \sqrt{6})$?

It will perhaps come as no surprise that it is possible to proceed to investigate p^3, or p^4, or p^k for any natural number k to see precisely what happens. However, the algebra involved in even finding and expanding p^3 is cumbersome, and attempting to explicitly calculate the fixed points of p^3 would mean having to solve a polynomial of degree eight. For this reason it would perhaps be more logical to start to investigate a more general case.

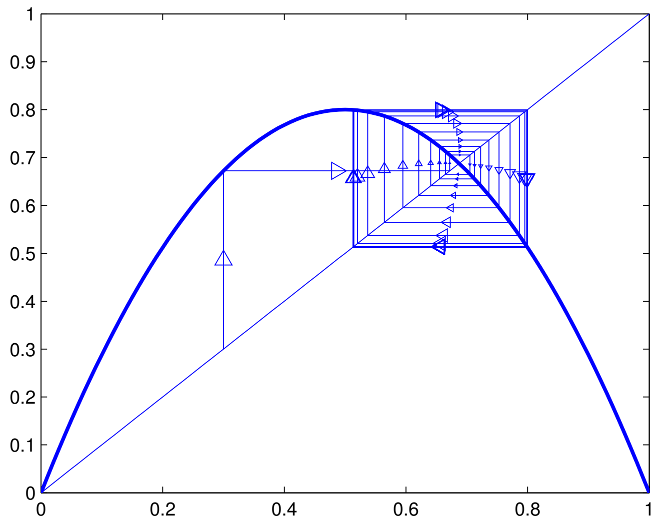

Figure 4: The cobweb diagram for $\lambda = 3.2$ and $x_0 = 0.3$, showing the first 50 iterates of x_0. [1]

3 Observing Order in a Population

When investigating the long-term behaviour of the population with the birth rate $\lambda \in (3, 1+\sqrt{6})$, it was found that p had two periodic points of period two. So then, the population x_n would start to alternate between two values as time progresses (by increasing the year n). It will perhaps come as no surprise that by investigating p^3 with certain values of λ, it would be found that new stable fixed points of p^3 are created that are periodic points of p with period three.

Finding the fixed points of p^k for any natural number k will always reduce down to having to solve a polynomial of some finite degree. Thus it seems logical that some values of λ can always be discovered such that, when using that value in the logistic map p, the solution will eventually become a finite sequence with period k. In other words, by using this particular model, it should always be possible to construct a population that will (eventually) oscillate between k values indefinitely $\forall k \in \mathbb{N}$.

As an example, I claim that with λ just beyond $1+2\sqrt{2}$ period three orbits are (eventually) created [2]. This claim won't be proved rigorously, but a justification of this fact will be given soon. The true significance of this fact will later be fully appreciated.

In fact, proving this claim would *automatically* prove that it would always be possible to construct a population that will (eventually) oscillate between k values indefinitely $\forall k \in \mathbb{N}$. This is a consequence of the following theorem:

Theorem 5 (Sarkovskii's Theorem (Special Case)). *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose f has a periodic point of period three. Then f has periodic points of all other periods [2]*

Proof: The proof will be omitted from this essay, however the full proof can be found in Devaney ([2]) pp. 60-62. □

Observent readers may note that the above theorem cannot be used in it's current form since the domain of the function p is $[0, 1]$ as opposed to $\mathbb{R}$. However, I claim that by drawing the graphs of $y = p$, $y = p^2$, $y = p^3$, ... and the line $y = x$ on the same graph, it will become clear that the two lines will never intersect outside of the domain $[0, 1]$, so the theorem must still apply to p despite the fact that the domain isn't restricted to the unit interval.

3.1 The Period Doubling Cascade

It has just been claimed that a period three orbit can be constructed, however that doesn't mean to say that a period three orbit is created when λ is increased through $1+\sqrt{6}$ (indeed I claimed that it doesn't). It is in fact p^4 that should be investigated to see what happens as λ is increased through $1+\sqrt{6}$.

It transpires that investigation is needed into the fixed points of p^2, p^4, p^8, p^{16}, etc. as λ is increased further. Below is a table showing precisely when orbits of certain periods are returned [4]:

Values of λ:	Orbit of period we get:
$(1, 3)$	1
$(3, 1+\sqrt{6})$	2
$(1+\sqrt{6}, 3.544090...)$	4
$(3.544090..., 3.564407...)$	8
$(3.564407..., 3.568750...)$	16
etc.	

This table shows an example of a period-doubling cascade: As λ is increased in p the number of periodic points of p is doubled.

3.2 The Feigenbaum Constant

The reader may have noticed that the intervals of λ in the table above rapidly become smaller and smaller. The rate of decrease is so rapid it begs the question: Will the widths of the intervals tend to 0 as the corresponding period tends to ∞?

The Feigenbaum constant $\delta \approx 4.6692016...$ is a famous constant that is the limit of successive ratios of the lengths of the intervals of λ from the table above. In other words, $lim_{n\to\infty}\left(\frac{d_n}{d_{n+1}}\right)$ exists in $\mathbb{R}$ and is equal to δ, where d_n is the length of the interval of values of λ corresponding to an orbit with period 2^n [3].

To see this actually happening, consider the sequence $(d_n)_{n=0}^{n=\infty}$ that starts off as (2, 0.449489..., 0.094601..., 0.020317..., 0.004343..., ...). From this it is easy to check that the first four terms of the sequence $\left(\frac{d_n}{d_{n+1}}\right)_{n=0}^{n=\infty}$ begin to converge to δ.

In fact, the Feigenbaum constant similarly exists for any one-dimensional map with a single parabolic maximum. Feigenbaum ironically only discovered that this constant exists because he didn't have very powerful computers to work with when he was investigating the logistic map[3] and wanted to cut down computing time. He tried to spot a pattern in the results he already had so that he could try and guess the value of λ at which the next period-doubling bifurcation would occur, which he used as the initial guess that the computer would then refine [3].

To answer the question posed at the start of this section, notice that as a consequence of the relationship $lim_{n\to\infty}\left(\frac{d_{n+1}}{d_n}\right) = \frac{1}{\delta} \in [0, 1)$ the sequence of

[3]and subsequently other maps such as the map $x \to k \sin x$

values of λ at which the successive period-doubling bifurcations occurs converges geometrically, and hence converges, to some real number.

3.3 The Map With Periodic Points of Period ∞

Using knowledge of the Feigenbaum constant it has been proved that is possible to find a value of λ such that, when exceeded and used in the logistic map, the population x_n will repeat with an "infinite" period. This λ is in fact approximately equal to 3.5699455... and is known as the accumulation point [4]. This is the last known point at which the long term behaviour of x_n can be easily observed and at which the long term behaviour can be at all predicted.

However it was also shown via Sarkovskii's Theorem that a λ can always be found so that the population x_n will oscillate between k values indefinitely $\forall k \in \mathbb{N}$. Therefore some form of order must exist beyond the accumulation point.

The graph in Figure 5, called an orbit diagram, shows the values that x_n will (eventually) orbit between as we iterate any arbitrary starting value x_0 by applying p. So, for example, when $\lambda \in (3, 1+\sqrt{6})$, the orbit diagram shows that x_n will orbit between two values.

Figure 5 clearly shows the period doubling cascade for values of $\lambda < 3.5699...$, by the fact that there are pitchfork bifurcations at values of λ that correspond to the values of λ discussed in the table in Section 3.1 (i.e. at $\lambda = 1$, $\lambda = 3$, $\lambda = 1+\sqrt{6}$, etc.). Figure 5 clearly also shows that around the value $\lambda = 3.5699...$ there is some kind of sudden change in the behaviour of the periodic points of p. This supports all that has been discussed thus far in this section.

Figure 6, which zooms in on Figure 5, shows an "island of stability" where more standard behaviour beyond $\lambda = 3.5699...$ is observed: At around $\lambda = 1+2\sqrt{2}$ it is clear that the erratic behaviour of the populations briefly settle down again. Moreover, the fact that there are three lines that 'develop' at around this point shows that x_n will (eventually) orbit between three values for such values of λ. This justifies my earlier claim that at values of λ just beyond $1+2\sqrt{2} \approx 3.828$ the map describes a population that will eventually oscillate between three populations indefinitely.

Beyond the accumulation point the graph is an example of a fractal, which loosely speaking is a pattern that has "infinite" detail; it is possible to zoom in on a part of the pattern so that a similar pattern can still be observed in just as clear detail as before. This explains why Figure 6 seems to show just as much detail as Figure 5.

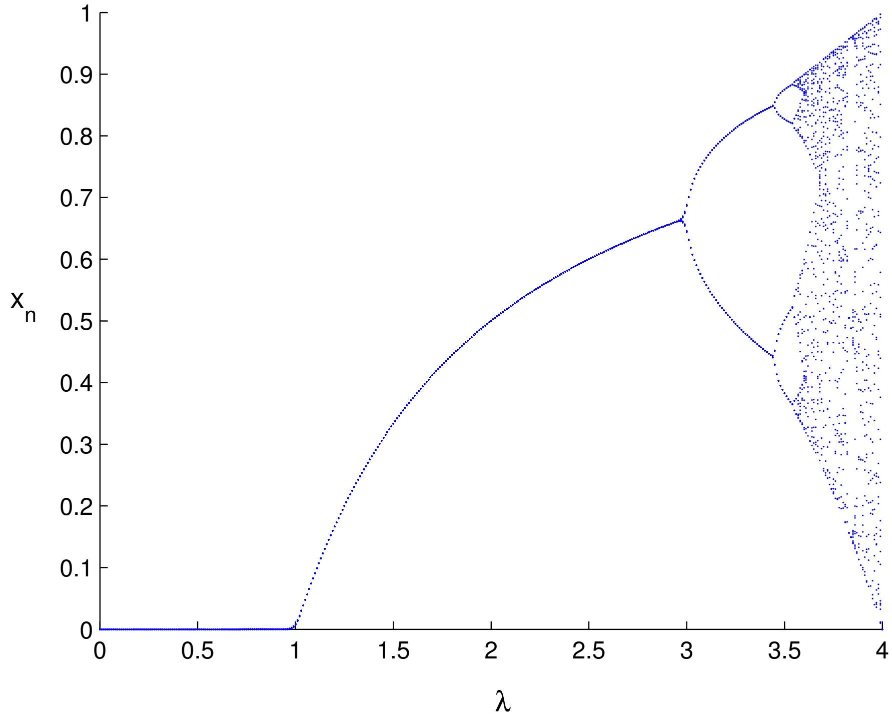

Figure 5: Orbit Diagram for $\lambda \in [0, 4]$ [1]

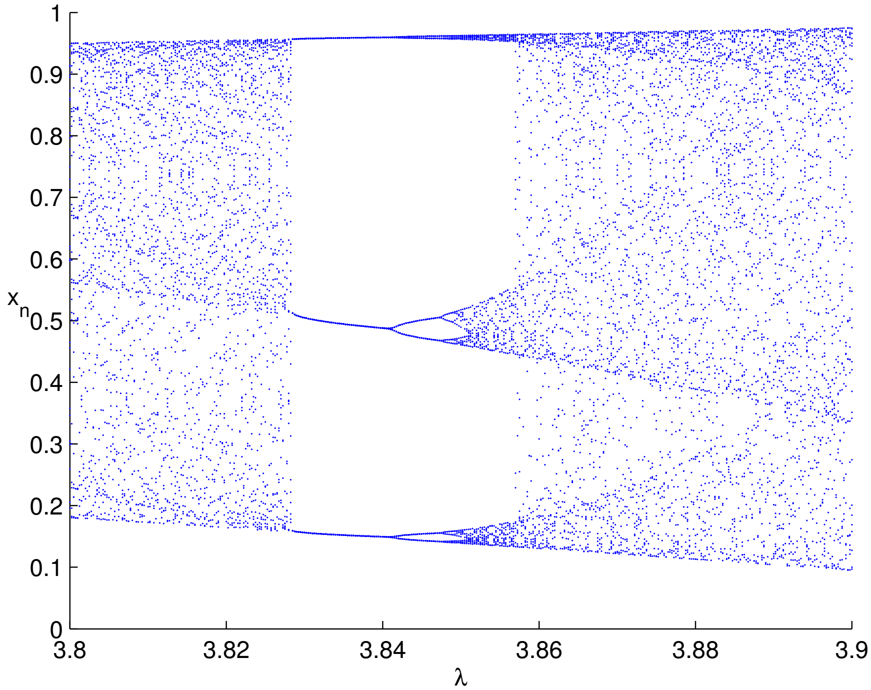

Figure 6: Orbit Diagram for $\lambda \in [3.8, 3.9]$ [1]

4 Observing Chaos in a Population

Firstly, the definition of what it means for a function to be chaotic is required ([2] page 50):

Definition 4 (Chaotic Function). Let V be a set. $f : V \to V$ is said to be chaotic on V if

1. f has sensitive dependence on initial conditions,
2. f is topologically transitive, and
3. Periodic points are dense in V.

Because the mathematics of chaos is very much beyond the scope of second year mathematicians, the precise definitions of what it means for f to have sensitive dependence on initial conditions, and of what it means for f to be topologically transitive, will be omitted.

Intuitively, a map f has senitive dependence on initial conditions at a point x if there exist points arbitrarily close to x that eventually separate from it by at least $\delta > 0$ under iteration of f ([2] page 49), and a map f is topologically transitive if it has points that eventually move under iteration from one arbitrarily small neighbourhood to any other ([2] page 49). A set U is said to be dense in V if the closure of U, the set of points x such that every neighbourhood of x meets U, is equal to V.

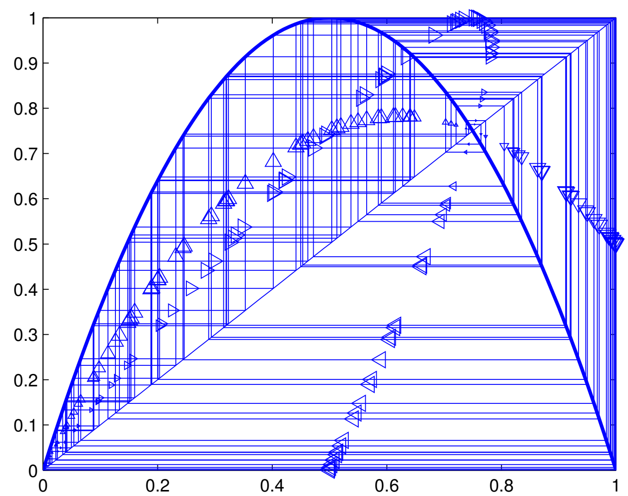

Figure 7: The cobweb diagram for $\lambda = 4$ and $x_0 = 0.2$, showing the first 100 iterates of x_0. [1]

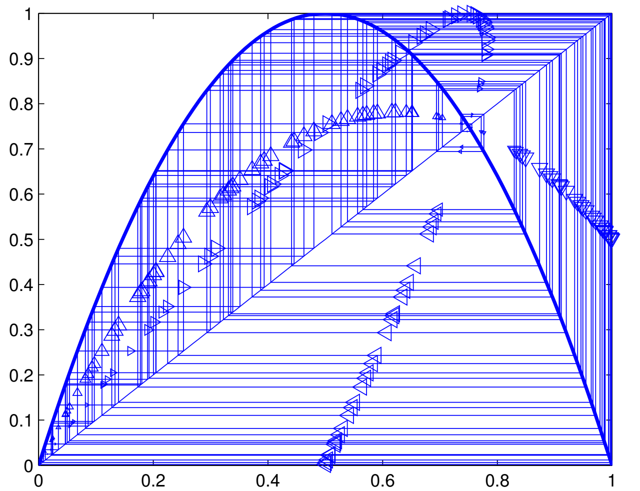

Figure 8: The cobweb diagram for $\lambda = 4$ and $x_0^* = 0.2005$, showing the first 100 iterates of x_0^*. [1]

At this point observe that, in Figure 5 (the orbit diagram for $\lambda \in [0, 4]$), the dots on the vertical line $\lambda = 4$ look as if they could be dense on $x_n \in [0, 1]$. In other words, for every point on the line, every neighbourhood of that point looks as though it could meet one of the dots. Moreover Figure 7, which is the by now familiar cobweb diagram with $\lambda = 4$, seems to show that p has points that eventually move under iteration from one arbitrarily small neighbourhood to any other.

Finally, comparing Figure 7 to Figure 8 which is precisely the same cobweb diagram except with a very slightly different initial condition $x_0^* = 0.2005$, shows that there exists at least one point extremely close to $x_0 = 0.2$ such that, when p is repeatedly applied to both starting conditions, the respective cobweb diagrams look completely different. In other words, points close to $x_0 = 0.2$ have sensitive dependence on initial conditions.

Of course this doesn't prove that the periodic points of p when $\lambda = 4$ actually are dense, or that p is topologically transitive and has sensitive dependence on initial conditions, however it could be an indication of the fact that p is a chaotic function[4] when $\lambda = 4$. In fact, it is a chaotic function:

Proposition 6. *The map $f(x) = 4x(1 - x)$ is chaotic on the interval [0,1]. In other words p, considered as the surjective function $p : [0, 1] \to [0, 1]$, is chaotic when $\lambda = 4$.*

[4] When considered as the surjective function $p : [0, 1] \to [0, 1]$

Proof: The proof will be omitted from this essay, however the full proof can be found in Example 8.9, Devaney ([2]) pp. 50-51. □

This Proposition conclusively proves that perturbing the value of the initial population x_0 even by a tiny amount will make the long-term behaviour of the solution vastly different, thus making the long-term behaviour seem almost "random".

By applying this established theory to population dynamics it has now been shown, using the logistic map as a model, that the behaviour of a single-species population will be far more complicated if the birth rate of the species is high. Moreover, as a consequence of Proposition 6, it has just been established that it would be practically impossible (with current technology, at least) to predict what the population of a species would be in a number of years time if it has the particular birth rate $\lambda = 4$.

5 Appendices and References

The following have been appended to this essay:

1. A detailed proof of the result $\left[p^2\left(\frac{(\lambda+1)\pm\sqrt{(\lambda+1)(\lambda-3)}}{2\lambda}\right)\right]' = -\lambda^2+2\lambda+4$, that I omitted from Proposition 4.
2. I created M-Files `p22832` and `p23` to produce Figures 2 and 3 respectively, which used the codes that have been appended.

References

[1] James C. Robinson: *An Introduction to Ordinary Differential Equations.* Cambridge University Press (2004).
I used the M-File `bifurcation.m` to produce Figures 5 and 6, and the M-file `logistic.m` to produce Figures 1 and 4, both of which are provided by Robinson for readers. They are available for download on the CUP website at:
`http://www.cambridge.org/uk/catalogue/catalogue.asp?isbn=9780521533911&resISBN13=9780521533911&parent=463&ss=res#resource`

[2] Robert L. Devaney: *An Introduction to Chaotic Dynamical Systems*, Second Edition. Addison-Wesley Publishing Company (1989).

[3] Ian Stewart: *Does God Play Dice?*, Second Edition. Penguin Books (1997).

[4] Weisstein, Eric W.: *"Logistic Map."*, `http://mathworld.wolfram.com/LogisticMap.html`. From MathWorld–A Wolfram Web Resource.