The Inverted Pendulum

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The motion of a standard pendulum has a single stable equilibrium at its lowest point. This is where the gravitational potential is at its minimum. However what is more surprising, is that if the midpoint is made to oscillate up and down quickly, there is also a stable point at the top! This system is referred to as a 'pendulum coupled to a harmonic oscillator'.
We will investigate the mechanics of this phenomena, and try to explain why and how it can occur.
In order to do this, we must first derive an equation for the motion of a pendulum under these conditions. This will require a bit of Hamiltonian mechanics:

1 The Hamiltonian System

1.1 Derivation

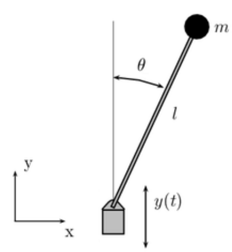

Figure 1: Pendulum model.

First we need the displacement of the pendulum from the centre:

$$s(t) = (lsin\theta, y + lcos\theta)$$

We then differentiate this with respect to time to find v:

$$v^2 = \dot{y}^2 - 2l\dot{\theta}\dot{y}sin\theta \ + l^2\dot{\theta}^2$$

Using the Lagrangian $L = T - V$ where T is the kinetic energy and V the potential gives:

$$L = \frac{1}{2}m(\dot{y}^2 - 2l\dot{\theta}\dot{y}sin\theta \ + l^2\dot{\theta}^2) - mg(y + lcos\theta) \tag{1}$$

Using the Euler-Lagrange Equation: $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta}$ with (1), and dividing through by ml, we get:

$$\frac{d}{dt}(-\dot{y}sin\theta + l\dot{\theta}) + \dot{\theta}\dot{y}cos\theta - gsin\theta = 0$$

Simplifying the above equation gives the required equation for the pendulum:

$$l\ddot{\theta} - \ddot{y}sin\theta = gsin\theta \tag{2}$$

1.2 Approximations

To make some calculations, we will assume that the acceleration, $\ddot{y}$, is modelled by a constant positive acceleration for the first half time period, and a constant negative acceleration of the same magnitue, for the second half. We will also assume that the oscillations are sufficiently small so that $sin\theta \approx \theta$. (see [1]):

$$l\ddot{\theta} - (g \pm A)\theta = 0 \quad \pm = \begin{cases} + & \text{for } 0 < t < \frac{T}{2}, \\ - & \text{for } \frac{T}{2} < t < T, \end{cases} \tag{3}$$

Lemma 1.1. *For a small time period T, fix D, the maximum displacement, then:*

$$D = \frac{1}{2}A\left(\frac{T}{4}\right)^2 \tag{4}$$

For this we have assumed that the pendulum is stable, and therefore the maximum displacement is at time $\frac{T}{4}$. Now we substitute (4) into equation (3) to obtain a high frequency approximation for the inverted pendulum:

$$l\ddot{\theta} - \left(g \pm \frac{32D}{T^2}\right)\theta \ = 0 \tag{5}$$

Now to show clearly that for high frequency, gravity is insignificant, we rescale equation (5), letting the new time period τ be 1. Then let $x(\tau)$ be the new equation of motion. So $\theta(t) = \theta(T\tau) \equiv x(\tau)$ as $\tau = \frac{t}{T}$ for $0 \leq t \leq T$. Now dividing through by l, and multiplying by T^2 gives:

$$T^2\ddot{\theta} - \left(\frac{gT^2}{l} \pm \frac{32D}{l}\right)\theta = 0$$

And by the chain rule, $x(\tau)'' = \theta(T\tau)'' = T^2\ddot{\theta}(T\tau) = T^2\ddot{\theta}(t)$. And so:

$$x'' - \left(\frac{gT^2}{l} \pm \frac{32D}{l}\right)x = 0 \pm = \begin{cases} + & \text{for } 0 < \tau < \frac{1}{2}, \\ - & \text{for } \frac{1}{2} < \tau < 1, \end{cases}$$

From the above equation, it is easy to see that for high frequency (i.e small T), the $\frac{gT^2}{l}$ term is small (and tends to 0, as T tends to 0), and so we can get a good approximation by considering the equation without this term. That is to say that we have the following (simplified) equation:

$$x'' \pm \omega^2 x = 0,\ \omega^2 = \frac{32D}{l},\ \pm = \begin{cases} + & \text{for } 0 < \tau < \frac{1}{2}, \\ - & \text{for } \frac{1}{2} < \tau < 1, \end{cases} \tag{6}$$

In the next section, we will look a bit more at the mechanical intuition behind the phenomena, before moving on to the topology of the pendulum. This section provides a bridge between this chapter and the next, and as such will be mainly definitions, and a couple of proofs at the end.

1.3 A look at stability

In the previous sections, we have shown that the equation of motion can be represented as a second order differential equation. Moving on from this, consider the motion as a matrix equation:

Proposition 1.2. *The equation of motion can be written in the form:*

$$\dot{\mathbf{x}} = A(t)\mathbf{x} \tag{7}$$

For some 2x2 matrix A with $A(t+T) = A(t)$ and $\mathbf{x} \in \mathbb{R}^2$.

Proof. For simplicity, we will write the acceleration, $\pm A$, as a single function $a = a(t)$. Let $\dot{\phi} = (-g-a)\theta$ and $l\dot{\theta} = \phi$, then $l\ddot{\theta} = \dot{\phi} = (-g-a)\theta$, which is the equation of motion. Therefore, write θ and ϕ in vector form:

$$\frac{d}{dt}\begin{pmatrix}\theta \\ \phi\end{pmatrix} = \begin{pmatrix}0 & \frac{1}{l} \\ -g-a & 0\end{pmatrix}\begin{pmatrix}\theta \\ \phi\end{pmatrix}$$

Or $\dot{\mathbf{x}} = A\mathbf{x}$ where, $A = \begin{pmatrix}0 & \frac{1}{l} \\ -g-a & 0\end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix}\theta \\ \phi\end{pmatrix}$. □

So we now know that the equation of motion can be written in matrix form, where A is a time-periodic matrix (as determined by a). The *fundamental matrix solution* to this equation is $\mathbf{x}(t) = e^{At}$. Before we work with this, we must show what it means for this solution to be *stable*.

Definition 1.3. *The system in Equation (7) is stable if every solution $\mathbf{x}(t)$ is bounded for all t.*

But this isn't very useful as it stands, because we want an easier way to determine if the solution is stable. To work towards this, write Equation (7) in the following form:

$$\dot{\mathbf{x}}(t) = X(t)\mathbf{x}(t) \tag{8}$$

To do this, simply consider the solutions $x_1(t)$, $x_2(t)$ where $x_1(0) = e_1$ and $x_2(0) = e_2$. Where e_1 and e_2 are the standard bases for $\mathbb{R}^2$. Then any solution x(t) can be written uniquely as a linear combination of these vectors. Specifically, if $x(t) = c_1x_1(t) + c_2x_2(t)$ and let X(t) be the matrix with columns $x_1(t)$ and $x_2(t)$, then:

$$\mathbf{x}(t) = X(t)\begin{pmatrix}c_1 \\ c_2\end{pmatrix} = X(t)\mathbf{x}(0)$$

Since $\mathbf{x}(0) = c_1x_1(0) + c_2x_2(0) = c_1e_1 + c_2e_2$.

Definition 1.4. *Now define the* Monodromy *matrix, M, (also called the Floquet matrix) for the above system by:*

$$M = X(T)$$

where T is the time period.

And finally define what it means for a matrix to be stable:

Definition 1.5. *A matrix M is* stable *if* $\exists C$ *such that* $|M^n| < C \ \forall n \in \mathbb{N}$.

Note:

- The above expression $|M^n|$ represents any matrix norm. In the same way as for vectors, there can be many types of norms used for matrices, but for the sake of argument, simply consider the *operator norm* which is, by definition:

$$|M| := \left\{ \frac{|M\mathbf{x}|}{|\mathbf{x}|} : \mathbf{x} \in \mathbb{R}^2, |\mathbf{x}| \text{ is the euclidean norm and } |\mathbf{x}| \neq 0 \right\}$$

Then a useful thereom would be to say that stability of the monodromy matrix implies stability of the system as a whole. Fortunately this is the case!

Theorem 1.6 (3)**.** *This system is stable if and only if its monodromy matrix is stable.*

Proof. Suppose the system is stable, then $|\mathbf{x}(t)| = |X(t)\mathbf{x}(0)|$ is bounded. So $\exists C$ such that $|X(t)| < C \ \forall t$. In particular, $\forall n \in \mathbb{N}$, $|M^n| = |X(n)| < C$ so M is stable.
Conversely, suppose that M is stable. Then $|X(n)| = |M^n| < C \ \forall n \in \mathbb{N}$. But for any time t, $X(t) = X(k + \tau)$ some $k \in \mathbb{N}$, $0 \leq \tau < 1$.
So $|X(t)| = |X(k + \tau)| = |X(k)X(\tau)| < C|X(\tau)|$. To see that $X(\tau)$ is bounded for $\tau \in [0, 1)$, note that X(t) is continuous for all t, and soo $|X(t)|$ is continuous for all t, and finite at X(0) and X(1). Therefore $\mathbf{x}(t) = X(t)\mathbf{x}(0)$ is bounded for all t. □

Finally, to finish this section, we will take a theorem from [1] showing exactly the conditions necessary to ensure stability, under high frequency approximations, as discussed in Section 1.2. This will use the fact that the system is stable if the eigenvalues are not real, which although intuitively true, is better discussed in the next chapter.

Theorem 1.7. *The high frequency inverted pendulum is stable when:*

$$\frac{D}{l} < \frac{\pi^2}{32}$$

Proof. Looking at Equation (6) and making the substitution $y = \frac{\dot{x}}{\omega}$, we can write the equation of motion as a system of equations as follows:

$$\begin{aligned} \dot{x} &= \omega y \\ \dot{y} &= \mp\omega x \end{aligned} \tag{9}$$

Then for the first half-time-period, the sign of ω is negative, and so we have circular motion i.e $\ddot{x} = \omega\dot{y} = -\omega^2 x$. And for the second half-time-period, the motion is hyperbolic, as we have $x^2 - y^2 = C$ for some constant C (This can be checked by differentiating and substituting). Then, in a geometric sense, it is sufficient to prove that the net total rotation is not an integer multiple of π i.e the overall motion is some kind of rotation. This would mean that there can be no real eigenvalues.

Now let V_R be the first motion, and V_H be the second. Then the rotational component of V at a point $z = (x, y)$, is $V \cdot \frac{z^\perp}{|z|}$, where $z^\perp = (y, -x)$. So:

$$\left| V_R \cdot \frac{z^\perp}{|z|} \right| = \frac{\omega(x^2 + y^2)}{|z|} \geq \omega |x^2 - y^2| = \left| V_H \cdot \frac{z^\perp}{|z|} \right|$$

With equality if and only if $x = 0$ or $y = 0$. Then the angle, α_H, turned due to V_H, is less than the angle of rotation, α_R, due to V_R, which is $\frac{\omega}{2}$. So the net rotation, $\alpha(z)$, is:

$$\alpha(z) = -\frac{\omega}{2} + \alpha_H(R(z))$$

Where the minus sign comes from the inital rotation being clockwise. And, in particular, as $\alpha_H(R(z)) \leq \alpha_R(z)$;

$$-\omega < \alpha(z) < 0$$

To get stability of the final solution, we need to ensure that $\alpha(z) \neq k\pi$ some integer k, so that the eigenvalues are imaginary. This means that we need:

$$|\omega| < \pi \iff \omega^2 < \pi^2 \iff \frac{32D}{l} < \pi^2$$

Then we have stability when $\frac{D}{l} < \frac{\pi^2}{32}$. $\square$

2 Symplectic group

Now that we have established the mechanics of the problem, we will investigate in more detail, the properties of the matrices which govern the motion of the pendulum.

2.1 Topology of Sp(2)

Firstly, we will show that the group of matrices belonging to Sp(2) are homeomorphic (and without proof, in fact diffeomorphic), to the open 3 dimensional region bounded by the 2- torus (for simplicity, we will call this region $\mathbb{T}$). For our purposes, it doesn't matter exactly what this means, except to say that we will end up with a continuous bijection between them, and that we can go back and forth between talking about matrices, and talking about points in $\mathbb{T}$, without any problem. First a few definitions:

Definition 2.1. *A 2 by 2 matrix A is* symplectic *if* $A^TJA = J$ *where* J *is the matrix:* $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Notes:

1. Using the matrix with the 1 and -1 switched is an equivalent definition.
2. A must be invertible, so we could also say that A $\in$ GL(2) the group of invertible 2 by 2 matrices

Definition 2.2. *The set Sp(2) :=* $\{A \in GL(2) \mid A^TJA = J\}$

Lemma 2.3. *Sp(2) is a group under standard matrix multiplication.*

Proof. Clearly $I_2 \in$Sp(2) and associativity is inherited from GL(2). And for any matrices A,B $\in$ Sp(2)

$$(AB)^TJ(AB) = B^T(A^TJA)B = B^TJB = J$$

So AB $\in$ Sp(2).
So just need to prove that if A $\in$ Sp(2) then $A^{-1} \in$ Sp(2): Let A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$
Then A^TJA = J if and only if:

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -(ad-bc) \\ ad-bc & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

So A $\in$ Sp(2) $\iff ad - bc = 1 \iff det(A) = 1$. Therefore, A $\in$ Sp(2) $\iff$ det(A) = 1 $\iff$ det(A^{-1}) = 1 $\iff A^{-1} \in$ Sp(2). □

Corollary 2.4. *Sp(2) ≡ SL(2), the special linear group.*

Now Given a matrix A $\in$ Sp(2), it is possible to decompose it into polar form: $A = PO$, where P is symmetric, positive definite, and O is orthogonal. (To see this, let P = $(AA^T)^{\frac{1}{2}}$, and O = $P^{-1}A$). Note that in this case, P $\in$ Sp(2) and O $\in$ SO(2).
The group of rotations can be thought of as a circle, where each point on the circle is associated to the orthogonal matrix with the respective rotation angle. So all that is required to show is the following:

Proposition 2.5. [2] *The group of all 2 by 2 symmetric, positive definite matrices with determinant 1 are homeomorphic to the open unit disk.*

Proof. If P is positive definite, then its eigenvalues must be positive. det(P) = 1 so the product of the eigenvalues is 1. So we get that $\lambda_1\lambda_2 = 1$ and $\lambda_1 > 0$ and $\lambda_2 > 0$ Therefore, $tr(A) = \lambda_1 + \lambda_2 \geq 2$, where tr(A) is the sum of the diagonal entries of P. Hence, without loss of generality, we can set tr(A) = 2coshτ some $\tau \geq 0$. As P is symmetric, we can write P as:

$$\begin{pmatrix} cosh\tau + a & b \\ b & cosh\tau - a \end{pmatrix}$$

some a,b $\in \mathbb{R}$. But det(A) = 1 = $cosh^2\tau - a^2 - b^2$. So

$$b^2 = cosh^2\tau - 1 - a^2 = sinh^2\tau - a^2$$

But this means that $|a| \leq |sinh\tau|$ So set a = cosσsinhτ some $\sigma \in \mathbb{R}$. This implies that $b^2 = sinh^2\tau(sin^2\sigma)$ so set $b = sin\sigma sinh\tau$ some $\sigma \in [0, 2\pi)$. Then:

$$P = \begin{pmatrix} cosh\tau + cos\sigma sinh\tau & sin\sigma sinh\tau \\ sin\sigma sinh\tau & cosh\tau - cos\sigma sinh\tau \end{pmatrix} \tag{10}$$

Now if we set $r = tanh^2\tau$ as in polar coordinates, and σ as the argument, we find that the group of such matrices P is homeomorphic to the unit disk
D = { z $\in \mathbb{C}$ | $|z| < 1$ } □

We now have the following theorem:

Theorem 2.6. *Sp(2) is homeomorphic to* $\mathbb{T}$.

Proof. Any matrix A $\in$ Sp(2) can be written uniquely in polar form: A = PO where O $\in$ SO(2), which is homeomorphic to the unit circle. And P is in the set of 2 by 2 symmetric, positive definite matrices with determinant 1, which is homeomorphic to the open unit disk. Conversely, for any such matrices P and O, $\exists A \in Sp(2)$ such that A = PO. So Sp(2) is homeomorphic to $\mathbb{R}/2\pi\mathbb{Z}$ x D. □

2.2 A quick look at the Eigenvalues

So given a symplectic matrix A, we know that the product of its eigenvalues must be 1, so let these be λ and $\frac{1}{\lambda}$. This gives the following lemma:

Lemma 2.7. *Given a matrix A with these eigenvalues,* $\lambda \in \mathbb{R} \cup U$ *where* $U = \{z \in \mathbb{C} \mid |z| = 1\}$.

Proof. Sufficient to check the case where lambda is not real, and so the discriminant of the following is negative:

$$det(A - \lambda I) = \lambda^2 - tr(A)\lambda + det(A) = \lambda^2 - tr(A)\lambda + 1 = 0$$

$$\iff \lambda = \frac{tr(A) \pm \sqrt{tr(A)^2 - 4}}{2} \implies \lambda = \frac{tr(A)}{2} \pm \frac{i\sqrt{4 - tr(A)^2}}{2}$$

Therefore if λ is one of these roots, then $\frac{1}{\lambda}$ is the other and so $\frac{1}{\lambda} = \bar{\lambda}$. This means that $\lambda\bar{\lambda} = 1 \iff |\lambda| = 1$. □

Now we have all that is needed to combine chapters 1 and 2. Before moving on, we will add the proposition that was missed out at the end of the previous chapter:

Proposition 2.8. *If the monodromy matrix has no real eigenvalues, then the system is stable.*

Proof. From Lemma 2.7, this equivalently says that the eigenvalues of the matrix lie on $U\backslash\{1,-1\}$. Given this matrix M, we can form a basis of eigenvectors, v_1 and v_2 with corresponding eigenvalues λ and $\bar{\lambda}$. Then given any vector x, we can write:

$$x = a_1v_1 + a_2v_2 \text{ for some a,b } \in \mathbb{R}$$

Then,

$$|M^n| = \frac{|M^n x|}{|x|} = \frac{|M^n(a_1v_1 + a_2v_2)|}{|x|} = \frac{|\lambda^n a_1v_1 + \bar{\lambda}^n a_2v_2|}{|x|} \leq \frac{|\lambda^n a_1v_1| + |\bar{\lambda}^n a_2v_2|}{|x|}$$

But λ and $\bar{\lambda}$ lie on the unit circle so $|\lambda^n| = |\bar{\lambda}^n| = 1$:

$$|M^n| \leq \frac{|a_1v_1| + |a_2v_2|}{|a_1v_1| + |a_2v_2|} = 1$$

And so $|M^n|$ is bounded for all $n \in \mathbb{N}$, and the system is stable. □

3 Bringing it all together

In the previous sections, we have shown how to model the pendulum as a matrix equation, and also discussed the topology of a specific group of matrices, $Sp(2)$. Firstly, we will show that this work hasn't been in vain, and that any solution matrix $X(t)$ where $\mathbf{x}(t) = X(t)\mathbf{x}(0)$, for any solution $\mathbf{x}(t)$, is sympectic. Fortunately, this is made easier by the fact that this group is exactly the same group as $SL(2)$ and so it is sufficient to show that the solution has determinant 1:

Lemma 3.1. *The matrix $X(t)$ has determinant 1 for all time t, and in particular, $X(t) \in Sp(2)$ $\forall t$.*

Proof. The matrix $X(t)$ satisfies the following differential equation:

$$\frac{d}{dt} det(X(t)) = tr(A) det(X(t)) = 0$$

Since $tr(A) = 0$. (For more on this, see [4]). Then by solving this equation we know that the determinat does not change with time, i.e $det(X) = C$ some $C \in \mathbb{C}$. Applying the initial condition; $X(0) = I \implies det(X(0)) = 1$, we get that C = 1, and so $X(t) \in Sp(2)$ $\forall t$ □

So from this, we can think of a solution at time t, as being a point in $\mathbb{T}$. But how does this show whether a given solution is stable? That is to say, can we determine whether a particular point in $\mathbb{T}$ represents a matrix with imaginary eigenvalues? [First lets 'divide' up $\mathbb{T}$ into real and complex eigenvalues.] Before we do this, it is important to make the following observation:
E*very matrix in* $Sp(2)$ *has two eigenvalues unique to that matrix:* λ *and* $\frac{1}{\lambda}$.

So instead of thinking of a matrix and then working out its eigenvalues, we can find a way of associating each point in $\mathbb{T}$, with one of the eigenvalues (and from this we will know both). We will do this by finding a continuous map (continuous because the eigenvalues depend continuously on X - see [3]) from Sp(2) to this set of eigenvalues. But how should be determine which eigenvalue to pick? If we are not consistent, then the map certainly won't be continuous. This is the basis for the first section:

3.1 The Krein-positive eigenvalue

In this section,

$$G := -iJ = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$\mathbb{U}$ is the unit circle in $\mathbb{C}$ about $\mathbf{0}$, and A is any matrix in $Sp(2)$.

Lets start straight away with a proposition:

Proposition 3.2. [2] *If A has complex eigenvalues and $\lambda \neq \pm 1$, then the eigenvectors, ξ and $\bar{\xi}$ of A are G-orthogonal with respect to the standard Hermitian product.*

Proof. Recall that $A^*JA = J \implies A^*GA = G$.

$$\langle G\xi, \bar{\xi}\rangle = \langle A^*GA\xi, \bar{\xi}\rangle = \langle GA\xi, A\bar{\xi}\rangle = \lambda^2\langle G\xi, \bar{\xi}\rangle$$

And so $\langle G\xi, \bar{\xi}\rangle = 0$. □

And in particular,

Corollary 3.3. *$\langle G\xi, \xi\rangle$ and $\langle G\bar{\xi}, \bar{\xi}\rangle$ are non-zero.*

So from this we know that they are either positive or negative. It is actually true that if one is positive then the other is negative (see [2]). So, to be consistent, make the following definition:

Definition 3.4. [2] *If $\lambda \in \mathbb{U}\backslash\{-1, 1\}$ is an eigenvalue of A, and ξ the corresponding eigenvector, then the Krein sign of λ is the sign of $\langle G\xi, \xi\rangle$. Specifically, if $\langle G\xi, \xi\rangle > 0$, λ is called the* Krein-positive *eigenvalue.*

Now that we have a way to pick given eigenvalues, define the following map:

Definition 3.5. *Define the rotation function* $\rho : Sp(2) \to \mathbb{U}$ *as:*

$$\rho(A) = \begin{cases} \lambda & \text{if } \lambda \in \mathbb{U}\backslash\{-1,1\} \text{ is the Krein-positive eigenvalue of } A \\ 1 & \text{if the eigenvalues of } A \text{ are real and positive} \\ -1 & \text{if the eigenvalues of } A \text{ are real and negative} \end{cases}$$

Then we can think of any point in $\mathbb{T}$ as being either: the Krein-positive eigenvalue of A or ± 1, depending on the eigenvalues of A.

All that this has shown is that it makes sense to think of a point, or points, inside $\mathbb{T}$ as being matrices with various eigenvalues. This turns out to be more useful when determining the stability of the hamiltonian system, which is what we are ultimately trying to achieve.
But first, here is, without proof, an interesting property of this function:

Proposition 3.6. $\rho(A^k) = \rho(A)^k$.

3.2 Subsets of $\mathbb{T}$

So given any matrix in Sp(2), we know that it either has complex eigenvalues on the unit circle, or real eigenvalues. So lets start by working out where they all lie inside $\mathbb{T}$:

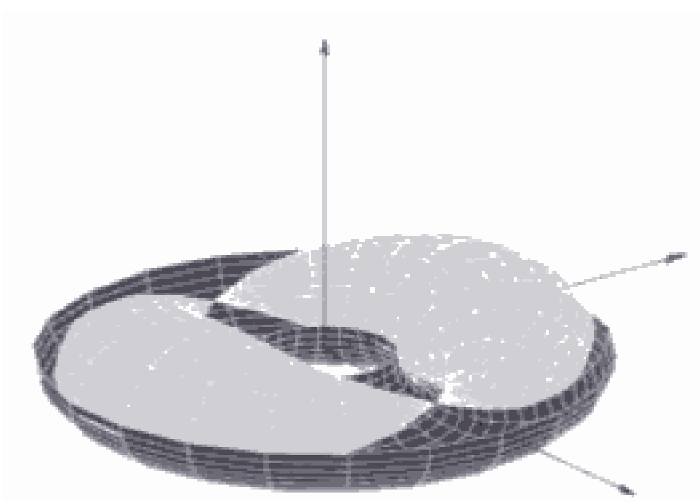

Figure 2: $\mathbb{T}$ with the region $r = sin^2\theta$ (Source: [2])

Proposition 3.7. *The set of matrices with eigenvalues* ± 1 *is the set* $\{(r, \theta, \sigma) | r = sin^2(\theta)\}$*, in toroidal co-ordinates (See fig.2).*

Proof. Using a standard identity:

$$det(A - \lambda I) = \lambda^2 - tr(A)\lambda + det(A) = \lambda^2 - tr(A)\lambda + 1 = 0$$

So A has a repeated eigenvalue, ± 1, if the discriminant of this quadratic is 0:

$$tr(A)^2 - 4 = 4cosh^2\tau cos^2\theta - 4$$

by substituting A = PO, (P from Equation (10)). And as we set $r = tanh^2\tau$, we have that:

$$cosh^2\tau = \frac{1}{1 - tanh^2\tau} = \frac{1}{1 - r}$$

And so

$$0 = 4(cosh^2\tau cos^2\theta - 1) \iff rcos^2\theta - 1 = 0 \iff r = sin^2\theta.$$

□

Which also yields a nice Corollary:

Corollary 3.8. *1. The set of matrices with eigenvalues* $\lambda \in \mathbb{U} \backslash \{-1, 1\}$ *is:*

$$\{(r, \theta, \sigma) | r < sin^2(\theta)\}$$

2. And the set of matrices with real eigenvalues $\lambda \neq \pm 1$ *is:*

$$\{(r, \theta, \sigma) | r > sin^2(\theta)\}$$

Proof. In case (1), the discriminant is negative. In case (2), the discriminant is positive. □

This means that we can split $\mathbb{T}$ into 2 sensible regions; real and non-real eigenvalues, with a boundary $r = sin^2\theta$ with eigenvalues ± 1 [see fig]. We know that the identity matrix has eigenvalues 1, and this corresponds to where the surface shrinks to a point on the right of Fig.2. Similarly $-I$ is the point on the left. Noting that the complete set $\mathbb{T}$ is *open*, we see that there is a 'gap' in the surface at its widest points, where $\theta = \pm\pi/2$. This is consistent with the continuity of the eigenvalues with respect to the parameters r, θ, σ. Putting this more precisely we have the following definition:

Definition 3.9. *Let* λ_0 *be the Krein-positive eigenvalue at* $(r_0, \theta_0, \sigma_0)$*, then the eigenvalues are continuous at* $(r_0, \theta_0, \sigma_0)$ *in* $\mathbb{T}$ *if:*
$\forall \epsilon_1, \epsilon_2, \epsilon_3 > 0$*, and* λ *the Krein-positive eigenvalue at* (r, θ, σ) *(also in* $\mathbb{T}$*),*
$\exists \delta > 0$ *such that* $|r - r_0| < \epsilon_1, |\theta - \theta_0| < \epsilon_2, |\sigma - \sigma_0| < \epsilon_3 \implies |\lambda - \lambda_0| < \delta$.

Therefore, the right-half surface, call α^+, (when $|\theta| < \pi/2$), are those matrices with eigenvalues +1, whereas the left-half surface, call α^-, (when $|\pi - \theta| < \pi/2$) are those with eigenvalues -1. Clearly these can not be 'touching', as this would break continuity, which is fortunate as the equation $r = sin^2\theta$ intersect $\mathbb{T}$, does not include the points where $\theta = \pm\pi/2$.
Geometrically, this means that the only (continuous) path from the eigenvalues +1 to eigenvalues -1, must pass through the inside of this surface, where the eigenvalues are complex. However, any path through complex eigenvalues can be thought of as moving (continuously) around the unit circle of Krein-positive eigenvalues. Although the proof of this should be topological, we will state this as a theorem, as it has very interesting consequences:

Theorem 3.10. *Any continuous path starting at a point on α^+ and finishing on α^- must contain (at least) every matrix with either:*

1. *Krein-positive eigenvalues with argument ≥ 0*
2. *Krein-positive eigenvalues with argument ≤ 0*

Simply put, this is because it must pass continuously around the unit circle starting at $(1, 0i)$ and finishing at $(-1, 0i)$. (However if the path winds completely around once, the eigenvalues can trace around the circle and can include other real eigenvalues).
This can be taken further, and some of this is discussed in [2], though is described in more detail than is required to explain stability of the pendulum.
To summarise, this section has shown that the imaginary eigenvalues lie inside the surface $\alpha^+ \cup \alpha^-$.

3.3 The Pendulum is stable when...

Suppose $\mathbf{x}(t) = X(t)\mathbf{x}(0)$ is the solution to the pendulum equation. Then by Section 1.3, the pendulum is stable:
$\Longleftrightarrow$ $\mathbf{x}(t)$ stays bounded for all t.
$\Longleftrightarrow$ the monodromy matrix, $M = X(T)$, where T is the time period, is stable.
$\Longleftrightarrow$ The monodromy matrix has eigenvalues on the unit circle.
But we know that X(t) is continuous with respect to t, and that $X(0) = I$. So X(0) is one of the points on the surface: $\alpha^+ \cup \alpha^-$. And as X(t) changes continuously, we get a path starting at this point and winding around $\mathbb{T}$, possibly passing in and out of these regions. If at time t=T, the endpoint of this path is inside one of these regions, then the monodromy matrix has complex eigenvalues, and so the pendulum equation is stable! This is the main result:

Theorem 3.11. *Let $X(t)$ for $0 \leq t \leq T$ be a path in $Sp(2)$, then the path starts at the Identity matrix. The pendulum equation is stable if and only if, the endpoint lies* inside *our surface. That is, $r < sin^2\theta$, in the polar matrix form.*

And this is it! Given the matrix equation for motion,we can now test whether the solution is stable, and so solved the problem.

An extension of this idea can be found in [2], and is called the *Maslov Index* for a given path. However, we will finish with the next chapter, which is more a corollary to the other chapters, and is an investigation into the sets of points inside our surface, corresponding to a particular eigenvalue. This has been included because it illustrates the point of Theorem 3.10, and is also very cool!

4 Extension: Eigenvalue Surfaces

For ease in this section, we will split our surface into four regions:

Definition 4.1. α^{++} *is the region bounded by the surface* α^{+} *for* $0 < \theta < \frac{\pi}{2}$.
α^{+-} *is the region bounded by the surface* α^{+} *for* $-\frac{\pi}{2} < \theta < 0$.
α^{-+} *is the region bounded by the surface* α^{-} *for* $\frac{\pi}{2} < \theta < \pi$.
α^{--} *is the region bounded by the surface* α^{-} *for* $-\pi < \theta < -\frac{\pi}{2}$.

We know from direct calculation that the surface $r = sin^2\theta$ corresponds to eigenvalues ± 1 but what about when the eigenvalues are, for example, $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$. Does this form a surface, or just seemingly random points inside $\alpha^{+} \cup \alpha^{-}$?
Knowing that passing from one point on α^{++} (say) to another passes continuously through the eigenvalues between, would suggest that they form surfaces (else we could change the path to 'miss' one!), or maybe the eigenvalues are densely packed inside!
As it turns out, the result is rather pretty, and the eigenvalues form smaller and smaller 'copies' of our region inside one another. The limit is the open disk at $\theta = \pm\frac{\pi}{2}$.

Also note visually, that it is not possible to pass from α^{++} to α^{-+} (and similarly for α^{+-} to α^{--}) without passing through every possible mini-region. Below are examples of surfaces generated by $\pm\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i$ in each of the 4 cases: (See appendix A for the MATLAB code, and Appendix B for some more pictures)

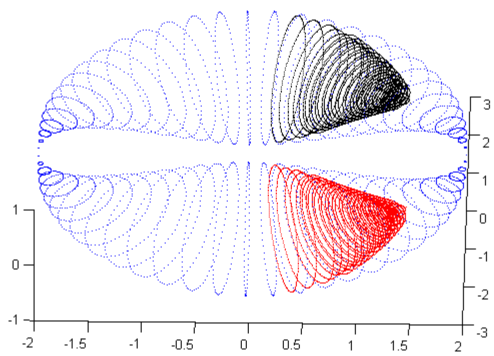

Figure 3: $\lambda = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ (black) and $\lambda = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$ (red).

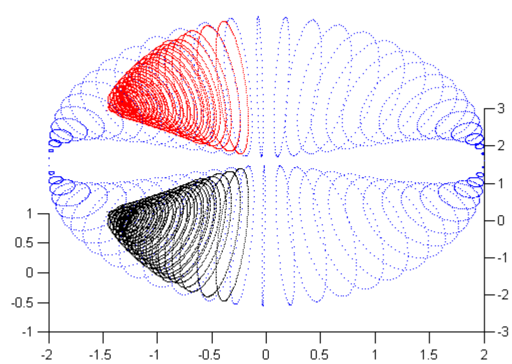

Figure 4: $\lambda = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ (red) and $\lambda = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$ (black).

Also, to show the 'nesting' of successive surfaces, see Fig.5 for a graph of our region, along with 2 surfaces; $\lambda = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$, and $\lambda = \frac{1}{\sqrt{3}} + \frac{\sqrt{2}}{\sqrt{3}}i$. (As discussed in earlier sections, the non-real eigenvalues must have absolute value 1).

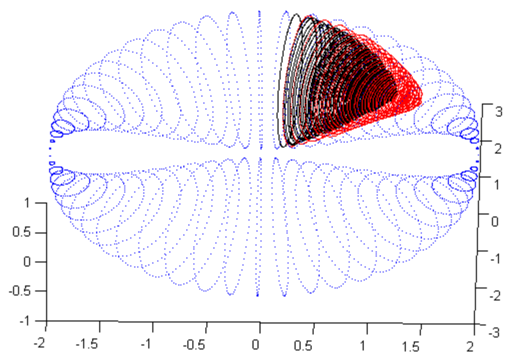

Figure 5: $\lambda = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ (red) and $\lambda = \frac{1}{\sqrt{3}} + \frac{\sqrt{2}}{\sqrt{3}}i$ (black).

Observations:

1. Pick a point on the α^{++} close to where it vanishes (see Fig.4). Then a vertical line passing from one side to the other will only pass through at most one of the surfaces.
   However, nearer to the limit disk, a vertical line will pass through both surfaces. This means that closer to the point, on any vertical line, the eigenvalues will pass continuously (in this case anticlockwise) around the unit circle, and then reach a maximum argument, and then return to $\lambda = 1$. In particular, each eigenvalue (apart from the one point in the centre) is passed through twice.

2. Any path starting on α^{++} and ending on α^{-+}, passes through every eigenvalue surface in α^{++}, then passes through the disk (in this case the disk: $\lambda = i$) (see Fig.6), and then every eigenvalue surface in α^{-+}. This is the statement of Theorem 3.10. This means that no matter how close to the surface (the torus) the path starts, and how close it stays to the surface when passing through the disk, the eigenvalues move continuously around the unit circle from 1 to -1.

3. Each of the eigenvalue surfaces tend to the boundary of $\mathbb{T}$ as $\theta \to \pm\pi/2$. If this wasn't the case, then observation 2 would be false (See Appendix B for a picture of 3 surfaces tending to the disk).

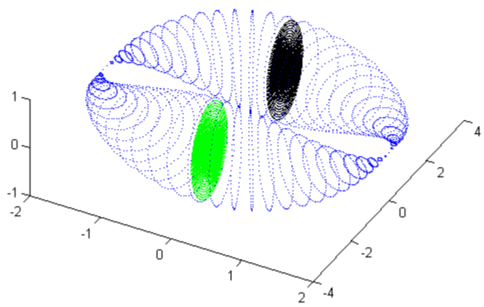

Figure 6: limit surfaces $\lambda = i$ (black) and $\lambda = -i$ (green).

4. Each eigenvalue on the unit circle (except ± 1) corresponds to one and only one surface inside one of the four sections (see Fig.4 and Fig.5 - note the symmetry of the surfaces). The surfaces have no boundary line sufficiently close to $\theta = \pm\pi/2$, because if this was the case, we could find a path "around" the surface (between it and the limit disk), thereby contradicting the above observations.

We conclude with the following very interesting observation:

When considering the pendulum equation, we know the matrices move continuously through $\mathbb{T}$*, and start at* I*. Then it must be the case that if the path winds round once and finishes back in the space of positive real eigenvalues, then for any non-real eigenvalue* λ *on the unit circle, there exists a time,* τ*, such that* $X(\tau)$ *has* λ *as its Krein-positive eigenvalue.*

5 References

[1] Stabilization of the invertend linearized pendulum by high frequency vibrations - Mark Levi and Warren Weckesser
[2] Morse theory for Hamiltonian systems - Alberto Abbondandolo
[3] Stability of the inverted pendulum - a topological explanation - Mark Levi
[4] Jacobi's formula for the derivative of a determinant - W. Kahan

6 Appendices

6.1 Appendix A: MATLAB CODE

Code calculating the rotation function:

```
function mas = maslov(r,theta,sigma)
i = sqrt(-1);
tau = atanh(sqrt(r));
% setting up the matrices
O(1,1) = cos(theta);
O(1,2) = -sin(theta);
O(2,1) = sin(theta);
O(2,2) = cos(theta);

   P(1,1) = cosh(tau) + cos(sigma)*sinh(tau);
P(1,2) = sin(sigma)*sinh(tau);
P(2,1) = sin(sigma)*sinh(tau);
P(2,2) = cosh(tau) - cos(sigma)*sinh(tau);
M = P*O;

   G(1,1) = 0;
G(2,2) = 0;
G(1,2) = i;
G(2,1) = -i;

   % setting up the eigenvalues and eigenvectors for the matrix [V,E] = eig(M);
val1 = E(1,1);
val2 = E(2,2);
vect1 = V(:,1);
vect2 = V(:,2);

   % finding the krein postivie eigenvalue
if real(val1) == abs(val1)
mas = val1; % if they are real
else % find the krein positive one
a = (G*vect1); % non-conjugate transpose
```

```
b = conj(vect1);
if a*b > 0
mas = val1;
else mas = val2;

end
```
———————————————-

Code which sketches the eigenvalue surface:

```
function out = sketch(value,reps,theps,colour,shrimp)
i = 0; % index for the array
epsilon = 1*10^(-6);
ep1 = 1/reps; % steps for r
ep2 = pi/theps; % steps for sigma and theta
points(1,1) = 0; % this is a storage array
hold on

for r = 0:ep1:1-epsilon;
for theta = 0:ep2:2*pi;
for sigma = 0:ep2:2*pi;
check = maslov(r,theta,sigma);
if abs(check - value) < 0.05
i = i+1;
points(i,1) = r;
points(i,2) = theta;
points(i,3) = sigma;
end
end
end
end

points;
% consider as in a torus radius outer radius: 3, inner radius: 1
for j = 1:i
rr = points(j,1);
tt = points(j,2);
ss = points(j,3);
length = 2 + rr*cos(ss);
X = length*cos(tt); % here is the X coordinate in Toroidal coords
Y = length*sin(tt); % here is the Y coord in Toroidal coords
Z = rr*sin(ss);
%plot(j,Z(j),'r+')

plot3(X,Y,Z,colour)
end
```

```
if shrimp == 'y' % don't need to plot every time
% This is a plot of the 'shrimp' to see how close to the 'shrimp' we get
for thetaall = 0:0.1:2*pi
for sigmaall = 0:0.1:2*pi
length = 2 +(sin(thetaall))^2*cos(sigmaall);
X = length*cos(thetaall);
Y = length*sin(thetaall);
Z = (sin(thetaall))^2*sin(sigmaall);
plot3(X,Y,Z,'b')
end
end
end
```

6.2 Appendix B: Additional Pictures

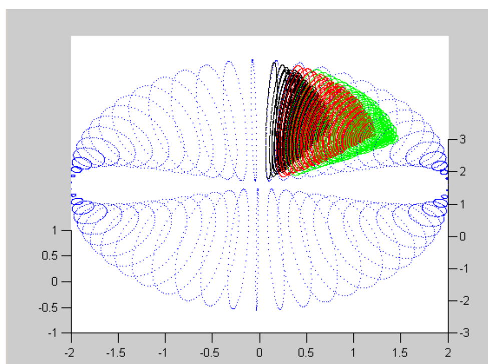

Figure 7: 3 surfaces where the eigenvalues are tending towards +i on the unit circle.

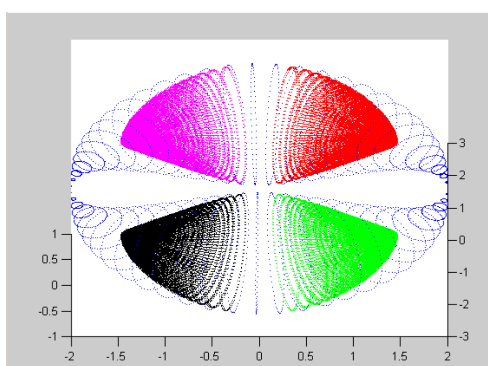

Figure 8: All 4 variations with extra resolution for θ close to $\pm\pi/2$.