Homology

Contents

1 Topological spaces and homotopy

1.1 CW complexes

A convenient way to consider topological spaces is to build them up using spaces we understand well. An obvious such candidate is the closed unit ball D^n in $\mathbb{R}^n$, also called an n-cell.

Definition 1.1 A cell complex X is constructed as follows:
Start with a discrete set X^0 of 0-cells (points with the discrete topology).
Inductively, construct the n-skeleton X^n of X by attaching n-cells D_i^n to X^{n-1} via maps of their boundaries $d_i\colon S_i^n \to X^{n-1}$; we write this $X^n = X^{n-1} \bigsqcup_{d_i} D_i^n$.[1]
We can either stop this construction after a finite number of steps n, in which case we set $X = X^n$, or we can continue indefinitely, and set $X = \bigcup_{n=0}^{\infty} X^n$, in which case we give X the weak topology: $A \subset X$ is open iff $A \cap X^n$ is open in X^n for all n.[2]

Cell complexes are also called CW complexes. CW stands for closure-finite weak: the closure of any cell only intersects with finitely many cells,[3] and the cell complex is given the weak topology.

1.2 Homotopy groups

1.2.1 The fundamental group

We want to look at the different paths we have in our topological space X; this might be a sphere or a torus for example, where we can intuitively notice different properties: a path on the sphere can always be deformed into a point, whereas on the torus that is not always true.
To make this more precise, a path is a continuous map $f\colon [0,1] \to X$. The idea of deforming paths is then the idea of homotopy:
A homotopy between two paths f and g is a continuous map $h\colon [0,1] \times [0,1] \to X$ such that $h(0,t) = f(t)$ and $h(1,t) = g(t)$. If such an h exists, we say that f and g are homotopic, and write $f \simeq g$. This allows us to say what we mean by being able to deform one path into another. We write $[g]$ for the equivalence class of g under the equivalence relation of homotopy, and call it the homotopy class of g.
We are also able to compose paths: given two paths f and g with $f(1) = g(0)$ we can compose the two paths to get fg which is $f(2t)$ for $0 \leqslant t \leqslant 1/2$ and $g(2t-1)$ for $1/2 \leqslant t \leqslant 1$.

[1] More precisely, for each map $d_i\colon S_i^n \to X^{n-1}$, we consider the disjoint union $D_i^n \coprod X^{n-1}$, and we then form the quotient space by identifying $x \in S_i^n$ to $d_i(x) \in X^{n-1}$. In category theory terms, X^n is the pushout of the following diagram: $\coprod_{i=1}^k D^n \hookleftarrow \coprod_{i=1}^k S^n \xrightarrow{\coprod_i d_i} X^{n-1}$.

[2] Categorically, X is the colimit of $X^0 \hookrightarrow X^1 \hookrightarrow X^2 \hookrightarrow \cdots$.

[3] For a proof of this fact, see [Hat02].

If we now consider closed paths starting and ending at some point $p \in X$ (ie, maps $S^1 \to X$), we have a group under composition of paths: the identity is just the constant path, and the inverse of any path is the same path with opposite orientation; associativity is clear. This allows us to consider the fundamental group $\pi_1(X)$ of our space, which is the group of homotopy classes of curves under the same group operation. We should really write $\pi_1(X, p)$ as this group is dependent on choice of basepoint, but if X is path-connected then we get the same answer as we can just consider the path from p to q and back which doesn't change anything.

The Seifert-Van Kampen Theorem

Given $X = A \cup B$, we would like to compute $\pi_1(X)$ given $\pi_1(A)$ and $\pi_1(B)$. For example, if A and B are two circles attached at a point, and X is their union, we can see that $\pi_1(X)$ is the free group on two generators, so that $\pi_1(X) \cong \pi_1(A) * \pi_1(B)$, where $*$ is the free product given by adding the presentations of A and B, so that $\langle a_1, \ldots, a_n | r_1, \ldots, r_n \rangle * \langle b_1, \ldots, b_n | s_1, \ldots, s_n \rangle = \langle a_1, \ldots, a_n, b_1, \ldots, b_n | r_1, \ldots, r_n, s_1, \ldots, s_n \rangle$. In general, the situation might be slightly more complicated, as the intersection of A and B might have nontrivial fundamental group, and we would be counting it twice if we just took the free product. Instead, we take the free product with amalgamation: $\pi_1(A \cap B)$ is a subgroup of both $\pi_1(A)$ and $\pi_1(B)$, so $\pi_1(X)$ is the free product of $\pi_1(A)$ and $\pi_1(B)$ with the added relations ab^{-1} for all a and b which are the image (in $\pi_1(A)$ and $\pi_1(B)$, respectively) of the same element in $\pi_1(A \cap B)$ under the inclusion maps. We write $\pi_1(X) \cong \pi_1(A) *_{\pi_1(A \cap B)} \pi_1(B)$. This is Van Kampen's Theorem:

Theorem 1.2 (Seifert-Van Kampen)
Let X, A and B be path connected spaces containing a given basepoint p such that $A \cup B = X$, $A \cap B$ is path connected and $p \in A \cap B$. Then $\pi_1(X) \cong \pi_1(A) *_{\pi_1(A \cap B)} \pi_1(B)$.[4]

Proof. Omitted, see [Hat02]. ❖

1.2.2 Higher dimensional homotopy groups

Following the idea of considering mappings $S^1 \to X$, we can consider mappings $S^n \to X$, and by a similar construction as for the fundamental group π_1 we obtain the homotopy groups π_n which are just the groups of mappings $S^n \to X$ quotiented out by homotopy equivalence. The group operation can be seen as follows: given $f, g \colon S^n \to X$, we get $fg \colon S^n \to X$ by the composite $S^n \to S^n \vee S^n \xrightarrow{f \vee g} X$ by collapsing S^n along the equator to two copies of S^n attached by a point (the wedge sum of S^n and S^n), and then mapping one copy with f and the other with g.
Some differences exist between π_1 and π_k for $k > 1$: for example, π_k are abelian, as we

[4]In the category of groups, we say that $\pi_1(X)$ is the pushout of $\pi_1(A) \hookleftarrow \pi_1(A \cap B) \hookrightarrow \pi_1(B)$.

can rotate the two spheres around, still keeping the basepoint fixed.[5]
The groups $\pi_k(S^n)$ behave much as we expect for $k < n$, because any map $S^k \to S^n$ is homotopic to a non-surjective map,[6] hence the image of S^k is contained in S^n with one point removed, which we can always contract to a point, so that $\pi_k(S^n) = 0$ for $k < n$. For $n = k$, the situation is analogous to the case $n = k = 1$, where $\pi_1(S^1) \cong \mathbb{Z}$, as all maps are determined up to homotopy by an integer (see Section 4.1), so that $\pi_n(S^n) \cong \mathbb{Z}$.
The trouble with homotopy groups really arises when we consider $\pi_k(S^n)$ with $k > n$: Taking a clue from Hopf, consider $S^3 \subset \mathbb{C}^2$ as $\{(z_1, z_2) : |z_1|^2+|z_2|^2 = 1\}$ and S^2 as the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Now define the Hopf fibration $f\colon S^3 \to S^2$ by $f\colon (z_1, z_2) \mapsto z_1/z_2$, which we can write in polar coordinates as $f(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$, with $r_1^2 + r_2^2 = 1$. As [Hat02] notices, if we fix a ratio $\rho = \frac{r_1}{r_2}$, we see that the set $\{(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})\}$ obtained by varying θ_1 and θ_2 forms a torus $T_\rho \subset S^3$. By varying ρ we fill up the whole of S^3 (accounting for the degenerate cases T_0 and T_∞). Noting that $f(\widetilde{z}_1, \widetilde{z}_2) = f(z_1, z_2)$ if and only if $(\widetilde{z}_1, \widetilde{z}_2) = (\lambda z_1, \lambda z_2)$ for some $\lambda \in \mathbb{C}$ (which then necessarily satisfies $|\lambda| = 1$), we see that $f^{-1}(p) \simeq S^1$. Hopf managed to show that this map $f\colon S^3 \to S^2$ isn't homotopic to the identity, so that $\pi_3(S^2) \neq 0$. In fact, this map generates $\pi_3(S^2)$, and $\pi_3(S^2) \cong \mathbb{Z}$.
Many other groups $\pi_k(S^n)$ for $k > n$ are non-trivial, and often quite tricky to compute; homotopy groups of spheres are currently not fully known, and this motivates the introduction of a more tractable algebraic invariant of topological spaces.

[5]It is possible to picture this by looking at the map the product gives from S^n to X which corresponds to one of the maps on each hemisphere with the equator shrunk down to the basepoint, and we can just rotate that sphere around, keeping basepoint fixed, interchanging the two (hemi)spheres. This was not possible with curves as we lacked an extra dimension to move the paths around on, so we could not move them round while still keeping the basepoint fixed.

[6]Showing this in fact requires some work, for example with π_1, if the map is surjective, take any point, and for each time the curve passes through that point, move that away slightly (via a homotopy), and repeat until that point is no longer attained. Another way to do this is using the Cellular Approximation Theorem (Theorem 2.22).

2 Homology

We have seen that homotopy is a very useful invariant of topological spaces, but often hard to compute: while $\pi_n(S^k)$ is 0 for $n < k$, it is not true that $\pi_n(S^k) = 0$ for $n > k$, the fundamental example being the Hopf fibration, as seen above. A solution is to introduce new groups H_n, called homology groups, which, like π_n, only depend on the $(n+1)$-skeleton of the space considered, but do not depend on the higher dimensional cells attached.[7] However, a few new concepts must be defined to be able to compute these groups.

2.1 Different homology theories

2.1.1 Simplicial homology

To construct simplicial homology, we first need the concept of simplicial complexes. An n-simplex is an n dimensional version of a triangle: a 2-simplex is a triangle, a 3-simplex is a tetrahedron, a 1-simplex is a line, a 0-simplex is a point, and so on. Each n-simplex contains its interior as it is an essentially n-dimensional object.
To be more precise, one can define the n-simplex as the set $\Delta^n = \{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1} : x_i \geqslant 0 \;\; \forall \;\; 0 \leqslant i \leqslant n \text{ and } \sum_{i=0}^n x_i = 1\}$. Given the basis vectors $e_0, \ldots, e_n$ of $\mathbb{R}^{n+1}$, we write this simplex as $[e_0, \ldots, e_n]$. In fact, this simplex has an orientation, given by the ordering $e_0, \ldots, e_n$ of its vertices. From now on, by n-simplex, we shall mean n-simplex with an orientation of its vertices.
In general, given an n-simplex s (specified by $n+1$ affine-linearly independent points $v_0, \ldots, v_n$), we have canonical "barycentric coordinates" given by the map $f\colon \Delta^n \to s$, $f\colon (x_0, \ldots, x_n) \mapsto \sum_{i=0}^n x_i v_i$. This shows that there is really only one n-simplex for each n, as there is an invertible affine transformation between all n-simplices for fixed n.
A simplicial complex is constructed by putting together simplices along their faces:

Definition 2.1 A simplicial complex S is a set of simplices with the following properties:
The intersection of any two simplices in S is a face of both simplices.
All faces of a simplex in S are also in S.

Remark 2.2 In particular, $\emptyset$ is a face of all simplices so is in S, which means that the first condition does make sense if the two simplices have empty intersection.

We can then define simplicial k-chains, which are formal sums of k-simplices $s_i \in S$, $\sum_i a_i s_i$ where the a_i are integer coefficients. The (abelian) group of sums of k-simplices under addition is denoted C_k. Note that $-s_i$ is taken to mean s_i with the opposite orientation of its vertices.
We would like to define a "boundary operator" $\partial_k\colon C_k \to C_{k-1}$; this boundary operator should depend on the orientation of the simplices. To have an idea of what pre-

[7] In fact, it will be the case that for spheres, $\pi_n(S^k) = H_n(S^k)$ for $1 \leqslant n \leqslant k$ and $H_n(S^k) = 0$ for $n > k$. See sections 2.2.1 and 2.2.2.

cisely this operator looks like, consider a triangle $s = [v_0, v_1, v_2]$, we want $\partial_2 s$ to be the cycle $[v_0, v_1] + [v_1, v_2] + [v_2, v_0]$, or $[v_1, v_2] - [v_0, v_2] + [v_0, v_1]$. So, similarly, for the standard n-simplex s, define $\partial_k s = \sum_{i=1}^{n}(-1)^i [v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n]$, also written $\sum_{i=1}^{n}(-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_n]$.

Theorem 2.3 The composition $\partial_{k-1} \circ \partial_k$ is identically 0.[8]

Proof. Consider $s = [v_0, \ldots, v_n]$, then $\partial_n s = \sum_{i=1}^{n}(-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_n]$ and

$$\partial_{n-1}\partial_n s = \sum_{j<i}(-1)^i(-1)^j [v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_i, \ldots, v_n] + \sum_{j>i}(-1)^i(-1)^{j-1} [v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_n]$$

where in the second term it is $(-1)^{j-1}$ because there are only $j-1$ vertices before v_j as $j > i$ and v_i was removed. Switching i and j in the second sum (as they are just dummy indices), we have that the two sums cancel as on one side we have $(-1)^{i+j}$ and on the other $(-1)^{i+j-1} = -(-1)^{i+j}$. Therefore $\partial_{k-1} \circ \partial_k = 0$. ❖

We can now form the so called chain complex[9]

$$\cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

The k-th homology group H_k is then defined to be the quotient group $\ker(\partial_k)/\mathrm{im}(\partial_{k+1})$.[10] Note that we can take coefficients in any abelian group instead of just the integers; in particular, we can notice that if we choose coefficients in a ring, we end up with the homology being a module over that ring.

Hence homology measures how far a sequence is from being exact:

A sequence $\cdots \to A \to B \to C \to \cdots$ is exact if at each element, the image of the incoming map is equal to the kernel of the next. The fact that a sequence is a chain complex tells us that the image is a subgroup of the kernel, and the homology measures how far $\ker(\partial_k) = \mathrm{im}(\partial_{k-1})$ is from being true.

A bit of terminology: elements of $\ker(\partial_n)$ are called cycles (the set of n-cycles is denoted Z_n), and elements of $\mathrm{im}(\partial_{n+1})$ are called boundaries (boundaries are elements of B_n). Then the homology group $H_n = Z_n/B_n$ measures how many n-cycles are not boundaries.[11]

[8]In fact, this can be seen geometrically: every $(k-2)$-dimensional simplex contained in a given k-simplex lies on exactly two faces of that k-simplex, which in this case guarantees that the composition vanishes.

[9]In general, a chain complex $C_\bullet = \{C_k, d\}$ is precisely this: a sequence of abelian groups (C_k) connected by an operator $d_k \colon C_k \to C_{k-1}$ such that $d \circ d = 0$.

[10]This is referred to as "taking the homology", as the same construction works for any chain complex.

[11]One might notice that if X is a single point, we have $\cdots \xrightarrow{\partial_1 = 0} C_0 \xrightarrow{\partial_0 = 0} 0$, so $\ker(\partial_0)/\mathrm{im}(\partial_1) \cong \mathbb{Z}$ when we could like this to be 0 instead. So we can define the reduced homology $\widetilde{H}_n$ as the homology of the complex $\cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{f} \mathbb{Z} = C_{-1} \xrightarrow{0} 0$, where f is the map $f(\sum_i a_i s_i) = \sum_i a_i$. Then as $f \circ \partial_1 = 0$, from the map $f \colon C_0 \to \mathbb{Z}$ we get a map $f \colon H_0 \to \mathbb{Z}$ with kernel $\widetilde{H}_0$ (by definition of $\widetilde{H}_0$) so $H_0 \cong \widetilde{H}_0 \oplus \mathbb{Z}$ and the other homology groups are unchanged. Note that we have to require $X \neq \emptyset$, as we would otherwise get $\widetilde{H}_{-1}(\emptyset) \cong \mathbb{Z}$.

As an example, consider the triangle without its interior as a simplicial complex S, we have $\partial_0 = 0$ so $\ker(\partial_0) \cong \mathbb{Z}^3$. Now, given a 1-chain $s = a_1[v_0, v_1] + a_2[v_1, v_2] + a_3[v_2, v_0]$, $\partial_1 s = (a_3 - a_1)[v_0] + (a_1 - a_2)[v_1] + (a_2 - a_3)[v_2]$. Letting $b_1 = a_3 - a_1$, $b_2 = a_1 - a_2$, $a_2 - a_3 = -(b_1 + b_2)$ so $\text{im}(\partial_1) \cong \mathbb{Z}^2$. From this calculation, $\partial_1(s) = 0 \leftrightarrow a_1 = a_2 = a_3$ so $\ker(\partial_1) \cong \mathbb{Z}$. There are no 2-chains so $\ker(\partial_2) \cong \text{im}(\partial_2) \cong 0$. Then $H_0(S) \cong \mathbb{Z}$, $H_1(S) \cong \mathbb{Z}$ and $H_2(S) \cong 0$.[12]In this case, it is important to notice $H_1(S) \cong \mathbb{Z}$ as it shows the 1-dimensional hole in the middle of the triangle.[13] This is precisely what makes cycles not boundaries: there is a hole in the middle so we cannot write the cycles as the boundary of something. In the case of the 2-simplex, we have patched together the hole in the middle and indeed the cycle going round the edges is precisely the boundary of the 2-simplex itself, as was computed.
The trouble with simplicial homology is that it is defined only for simplicial complexes, but we would like to be able to carry out the same calculations for any topological space, this idea leads to the concept of singular homology.

2.1.2 Singular homology

The idea of singular homology is to consider maps from simplices into the topological space instead of the simplices themselves.

Definition 2.4 A singular n-simplex on a topological space X is a continuous map $\sigma\colon \Delta^n \to X$ from the standard n-simplex to X.[14]

We can then define the boundary operator in a similar fashion than for simplicial homology. Given a (singular) n-simplex $\sigma|[v_0, \dots, v_n]$, we have $\partial_n \sigma = \sum_{i=1}^{n} (-1)^i \sigma|[v_0, \dots, \hat{v_i}, \dots, v_n]$ where $[v_0, \dots, \hat{v_i}, \dots, v_n]$ is identified with Δ^{n-1} so that $\partial_n \sigma$ becomes a $(n-1)$-simplex. We can then define the chain complex $C^S_\bullet$, and we still have the notions of cycles and boundaries. We also have that $\partial_{k-1} \circ \partial_k = 0$, with essentially the same proof as before. The trouble is that the spaces we are considering are "much larger" than before, as we are considering all the possible maps σ from Δ^n to X, and the homology $H^S_n(X) = \ker(\partial_k)/\text{im}(\partial_{k+1})$ is a quotient of two very large groups. We will soon see that this isn't a problem.
When we have two spaces X and Y related by a map $f\colon X \to Y$, we would like to know how the homology groups are related. The idea of relating the homology groups through a map $f\colon X \to Y$ is encapsulated in the following definition:

Definition 2.5 A chain map between two chain complexes $C_\bullet(X)$ and $C_\bullet(Y)$ is a col-

[12]Using reduced homology, we know that $\widetilde{H}_0(S) \cong 0$, $\widetilde{H}_1(S) \cong \mathbb{Z}$, $\widetilde{H}_2(S) \cong 0$, perhaps a "more accurate measure of the holes of S".

[13]One might think this hole is 2-dimensional, but what we are measuring is how 1-cycles fail to be boundaries, so we are measuring some 1-dimensional phenomenon.

[14]This is where the world "singular" comes from: it is not required that σ be injective, for example, and there might be many "singularities" in the image of σ.

lection of homomorphisms $\{f_i\}$ such that the following diagram commutes:

$$\begin{array}{ccccccc} \cdots \xrightarrow{\partial_{k+1}} & C_k(X) & \xrightarrow{\partial_k} & C_{k-1}(X) & \xrightarrow{\partial_{k-1}} \cdots \\ & \downarrow{f_k} & & \downarrow{f_{k-1}} & \\ \cdots \xrightarrow{d_{k+1}} & C_k(Y) & \xrightarrow{d_k} & C_{k-1}(Y) & \xrightarrow{d_{k-1}} \cdots \end{array}$$

Proposition 2.6 A chain map induces homomorphisms in homology.

Proof. It suffices to notice that the f_k take cycles to cycles (as $f_k(0) = 0$) and boundaries to boundaries. ❖

Proposition 2.7 A map $f\colon X \to Y$ induces a chain map from the singular chain complex of X to that of Y.

Proof. If we have a map $f\colon X \to Y$, a singular simplex $\sigma\colon \Delta^k \to X$ induces a singular simplex $f_*(\sigma) = (f \circ \sigma)\colon \Delta^k \to Y$. We just need to show the above diagram commutes, ie that $f_*(\partial(\sigma)) = \partial(f_*(\sigma))$. As $f_*\left(\sum_i a_i\sigma_i\right) = \sum_i a_i f_*(\sigma_i)$, we have that:

$$\begin{aligned} f_*(\partial(\sigma)) &= f_*\left(\sum_i (-1)^i \sigma|[v_0, \ldots, \hat{v}_i, \ldots, v_n]\right) \\ &= \sum_i (-1)^i f_*(\sigma|[v_0, \ldots, \hat{v}_i, \ldots, v_n]) = \partial(f_*(\sigma)) \end{aligned}$$ ❖

Notice that if we have $X \xrightarrow{f} Y \xrightarrow{g} Z$ then $(g \circ f)_* = g_* \circ f_*$ as $(g \circ f)_*(\sigma) = (g \circ f) \circ \sigma = g \circ (f \circ \sigma) = (g_* \circ f_*)(\sigma)$. It is also clear that $\mathrm{id}_* = \mathrm{id}$. [15]

Theorem 2.8 If two maps $f, g\colon X \to Y$ are homotopic, the homomorphisms induced in homology by f and g are the same.

Proof. Following [Hat02], consider the homotopy $F\colon X \times I \to Y$ between f and g. We can subdivide the product $\Delta^n \times I$ into $(n+1)$-simplices as follows. Take $\Delta^n \times \{0\} = [v_0, \ldots, v_n]$ and $\Delta^n \times \{1\} = [w_0, \ldots, w_n]$. Then the n-simplex $[v_0, \ldots, v_i, w_{i+1}, \ldots, w_n]$ is the graph of $\varphi_i\colon \Delta^n \to I$ defined by $\varphi_i(t_0, \ldots, t_n) = t_{i+1}, \ldots, t_n$. $\varphi_i \leqslant \varphi_{i-1}$, and the region between the graphs of φ_1 and φ_{i-1} is $[v_0, \ldots, v_i, w_i, \ldots, w_n]$. Then, because $0 = \varphi_n \leqslant \varphi_{n-1} \leqslant \cdots \leqslant \varphi_0 \leqslant \varphi_{-1} = 1$, we see that $\Delta^n \times I$ is the union over i of all $(n+1)$-simplices $[v_0, \ldots, v_i, w_i, \ldots, w_n]$.
Now, for $\sigma\colon \Delta^n \to X$, we define the prism operators $P\colon C_n(X) \to C_{n+1}(Y)$ by

$$P(\sigma) = \sum_i (-1)^i F \circ (\sigma \times \mathrm{id})|[v_0, \ldots, v_i, w_i, \ldots, w_n]$$

[15]This means that singular homology is a functor, from the category of topological spaces to the category of abelian groups.

where $F \circ (\sigma \times \mathrm{id})$ is the composition $\Delta^n \times I \to X \times I \to Y$. Then

$$\begin{aligned}\partial P = &\sum_{j \leqslant i} (-1)^i (-1)^j F \circ (\sigma \times \mathrm{id})|[v_0, \ldots, \hat{v}_j, \ldots, v_i, w_i, \ldots, w_n] \\ &+ \sum_{j \geqslant i} (-1)^i (-1)^{j+1} F \circ (\sigma \times \mathrm{id})|[v_0, \ldots, v_i, w_i, \ldots, \hat{w}_j, \ldots, w_n]\end{aligned}$$

so that the terms in the two sums cancel when $i = j$ except for $F \circ (\sigma \times \mathrm{id})|[\hat{v}_0, w_0, \ldots, w_n]$ which is $g_* \circ \sigma$ and $F \circ (\sigma \times \mathrm{id})|[v_0, \ldots, v_n, \hat{w}_n]$ which is $-f_* \circ \sigma$. When $i \neq j$, we are left with exactly $-P\partial(\sigma)$. Hence $\partial P + P\partial = g - f$.
Now, if $\alpha \in C_n(X)$ is a cycle, $g_*(\alpha) - f_*(\alpha) = \partial P(\alpha)$ which is a boundary, so that $g_*(\alpha)$ and $f_*(\alpha)$ determine the same homology class. Hence the theorem is proven. ❖

Corollary 2.9 If X and Y are homotopy equivalent, then all their homology groups coincide.

Proof. X and Y are homotopy equivalent means there exists $f \colon X \to Y$ and $g \colon Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.
We then have that $(g \circ f)_* = g_* \circ f_* = \mathrm{id}_* = f_* \circ g_* = (f \circ g)_*$, where id_* is just the identity on the homology groups. Hence g_* and f_* are inverses of each other so are isomorphisms. ❖

Definition 2.10 Two chain maps $\{f_i \colon C_i(X) \to C_i(Y)\}$ and $\{g_i \colon C_i(X) \to C_i(Y)\}$ are chain-homotopic if there exists maps $\{h_i \colon C_i(X) \to C_{i+1}(Y)\}$ such that $f - g = dh + hd$. Equivalently, given the diagram

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d} & C_{k+1}(X) & \xrightarrow{d} & C_k(X) & \xrightarrow{d} & C_{k-1}(X) & \xrightarrow{d} & \cdots \\ & & f\downarrow\downarrow g & & f\downarrow\downarrow g & & f\downarrow\downarrow g & & \\ \cdots & \xrightarrow{d} & C_{k+1}(Y) & \xrightarrow{d} & C_k(Y) & \xrightarrow{d} & C_{k-1}(Y) & \xrightarrow{d} & \cdots \end{array}$$

h must make the following diagram commute

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d} & C_{k+1}(X) & \xrightarrow{d} & C_k(X) & \xrightarrow{d} & C_{k-1}(X) & \xrightarrow{d} & \cdots \\ & \swarrow h & \downarrow f-g & \swarrow h & \downarrow f-g & \swarrow h & \downarrow f-g & \swarrow h & \\ \cdots & \xrightarrow{d} & C_{k+1}(Y) & \xrightarrow{d} & C_k(Y) & \xrightarrow{d} & C_{k-1}(Y) & \xrightarrow{d} & \cdots \end{array}$$

Remark 2.11 Notice that our proof of Theorem 2.8 implies that two chain-homotopic chain maps induce the same homomorphisms in homology. In particular, if a chain map is chain-homotopic to the identity, it induces an isomorphism in homology.

Given a space X and a subspace A,[16] we might want to consider the homology of X modulo the homology of A. Relative homology makes this possible:

[16]We write (X, A) for the inclusion $A \hookrightarrow X$, and call it a pair.

Definition 2.12 Define $C_k(X, A) = C_k(X)/C_k(A)$. The chain complex $C_\bullet(X)$ induces a chain complex $C_\bullet(X, A)$, whose homology $H_k(X, A)$ is called the k-th relative homology group.

In this case, we notice that cycles of the relative homology (also called relative cycles) are the elements $c \in C_k(X)$ such that $\partial c \in C_{k-1}(A)$, and relative boundaries are elements $c \in C_k(X)$ that can be written as $\partial a + b$ with $a \in C_{k+1}(X)$ and $b \in C_k(A)$.
Notice we can form an exact sequence $0 \to C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{p} C_n(X)/C_n(A) \to 0$ where i is inclusion and p is the natural quotient map. This leads us to the following commutative diagram

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{n+1}(A) & \xrightarrow{i} & C_{n+1}(X) & \xrightarrow{p} & C_{n+1}(X)/C_{n+1}(A) & \longrightarrow & 0 \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\
0 & \longrightarrow & C_n(A) & \xrightarrow{i} & C_n(X) & \xrightarrow{p} & C_n(X)/C_{n+1}(A) & \longrightarrow & 0 \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\
0 & \longrightarrow & C_{n-1}(A) & \xrightarrow{i} & C_{n-1}(X) & \xrightarrow{p} & C_{n-1}(X)/C_{n+1}(A) & \longrightarrow & 0 \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

This diagram commutes because ∂ is the same in each column, so restricts to the other columns by acting in the same way. This commutative diagram is called a short exact sequence of chain complexes.

Lemma 2.13 (Zigzag lemma)
Suppose we have the following "short exact sequence of chain complexes" (of abelian groups):[17]

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_{n+1} & \xrightarrow{a_{n+1}} & B_{n+1} & \xrightarrow{b_{n+1}} & C_{n+1} & \longrightarrow & 0 \\
 & & \downarrow \partial_{a,n+1} & & \downarrow \partial_{b,n+1} & & \downarrow \partial_{c,n+1} & & \\
0 & \longrightarrow & A_n & \xrightarrow{a_n} & B_n & \xrightarrow{b_n} & C_n & \longrightarrow & 0 \\
 & & \downarrow \partial_{a,n} & & \downarrow \partial_{b,n} & & \downarrow \partial_{c,n} & & \\
0 & \longrightarrow & A_{n-1} & \xrightarrow{a_{n-1}} & B_{n-1} & \xrightarrow{b_{n-1}} & C_{n-1} & \longrightarrow & 0 \\
 & & \downarrow \partial_{a,n-1} & & \downarrow \partial_{b,n-1} & & \downarrow \partial_{c,n} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

[17] In fact, this holds in any abelian category, not only the category of abelian groups.

There is then a family of "connecting homomorphisms" d_i for homology:

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
H_{n+1}(A) & \xrightarrow{(a_{n+1})_*} & H_{n+1}(B) & \xrightarrow{(b_{n+1})_*} & H_{n+1}(C) \xrightarrow{d_{n+1}} \\
H_n(A) & \xrightarrow{(a_n)_*} & H_n(B) & \xrightarrow{(b_n)_*} & H_n(C) \xrightarrow{d_n} \\
H_{n-1}(A) & \xrightarrow{(a_{n-1})_*} & H_{n-1}(B) & \xrightarrow{(b_{n-1})_*} & H_{n-1}(C) \\
\vdots & & \vdots & & \vdots
\end{array}
$$

so that we get the following long exact sequence:

$$\cdots \to H_{n+1}(C) \overset{d_{n+1}}{\to} H_n(A) \overset{(a_n)_*}{\to} H_n(B) \overset{(b_n)_*}{\to} H_n(C) \overset{d_n}{\to} \cdots \overset{(b_0)_*}{\to} H_0(C) \to 0$$

Proof. Omitted, see [Mac98]. ❖

Proposition 2.14 The relative homology groups fit in the following long exact sequence:

$$H_{n+1}(X,A) \overset{d_{n+1}}{\to} H_n(A) \overset{i_*}{\to} H_n(X) \overset{p_*}{\to} H_n(X,A) \overset{d_n}{\to} \cdots \overset{p_*}{\to} H_0(X,A) \to 0$$

Proof. This follows directly from the Zigzag lemma, considering the diagram we had for relative chains. ❖

Notice that $H_n(X,A)$ is defined as the homology of $C_n(X)/C_n(A)$, it is not in general equal to $H_n(X/A)$. However $H_n(X,A) \cong H_n(X/A)$ in the case that A is a deformation retract of some neighbourhood U of itself in X, which is a homotopy between the identity map on U and a continuous map $U \to A$ which is the identity on A, called a retract. An important example of this is when X is a CW complex and A is a subcomplex. For a more detailed justification, see [Hat02].

Another important property of homology groups is that of excision: excision corresponds to the simple idea that, for the relative homology groups $H_k(X,A)$, we should be able to remove a subspace S of A without changing the relative homology.

Theorem 2.15 (Excision)

If the closure of S is contained in the interior of A, then the inclusion $i\colon (X-S, A-S) \to (X,A)$ induces isomorphisms $i_*\colon H_n(X-S, A-S) \to H_n(X,A)$

Proof. This theorem can be restated as follows: Write $B = X \setminus S$ so $S = X \setminus B$. We then have $A \cap B = A \setminus S$. We know that $\mathrm{cl}(S) \subset \mathrm{int}(A)$, so as $X \setminus \mathrm{int}(B) = \mathrm{cl}(Z)$, this is equivalent to $X = \text{int}(A) \cup \text{int}(B)$. The theorem then becomes $H_n(B, A\cap B) \cong H_n(X, A)$. To prove the theorem, we use a lemma. Let $\mathcal{U} = \{U_i\}$ be a collection of subspaces of X whose interiors cover X. Define $C_n^{\mathcal{U}}(X)$ as the subgroup of $C_n(X)$ consisting of chains of the form $\sum_i a_i\sigma_i$ for $\sigma_i \in C_n(U_i)$. We write the homology of this complex as $H_n^{\mathcal{U}}(X)$.

Lemma 2.16 Let i be the inclusion $i\colon C_n^{\mathcal{U}}(X) \to C_n(X)$. There exists a chain map $p\colon C_n(X) \to C_n^{\mathcal{U}}(X)$ such that $p \circ i = \text{id}$ and that $i \circ p$ is chain homotopic to the identity.

Proof. See [Hat02], the proof involves a technique known as barycentric subdivision, which allows to decompose a simplex by cutting it up into smaller pieces, until we can fully decompose a simplex in X to a sum of simplices contained in the U_i. This is, in essence, what the map p does. ❖

Remark 2.17 Notice that this lemma implies $H_n^{\mathcal{U}}(X) \cong H_n(X)$ for all n by Remark 2.11.

In this case, we have $\mathcal{U} = \{A, B\}$. Write $C_n(A + B)$ for $C_n^{\mathcal{U}}(X)$. We know $p \circ i = \text{id}$ and $\partial h + h\partial = \text{id} - i \circ p$ for some h as $i \circ p$ is chain homotopic to the identity. Hence p and i induce isomorphisms on homology $i_*\colon H_n(A + B) \to H_n(X)$, $p_*\colon H_n(X) \to H_n(A + B)$. But i takes chains in A to chains in A (as it is just inclusion) and therefore so does p. Therefore $i\colon C_n(A + B)/C_n(A) \to C_n(X)/C_n(A)$ induces an isomorphism in homology $i_*\colon H_n^{\mathcal{U}}(X, A) \to H_n(X, A)$. We also have the inclusion $j\colon C_n(B)/C_n(A\cap B) \to C_n(A + B)/C_n(A)$, which must be an isomorphism as both sides consist exactly of the chains in B with no cycles in A. So $H_n(B, A \cap B) \cong H_n^{\mathcal{U}}(X, A)$, hence $H_n(X, A) \cong H_n(B, A \cap B)$. ❖

Also note that given any chain complex $C_\bullet = \{C_k, \partial\}$ we can form its dual, which is a cochain complex[18] $C^\bullet = \left\{C^k, \delta\right\}$ by letting $C^k = \text{Hom}(C_k, \mathbb{Z})$. Given $f_k\colon C_k \to \mathbb{Z}$, we want to find $\delta(f_k)\colon C_{k+1} \to \mathbb{Z}$. As f_k only operates on elements of C_k, we can define $\delta(f_k)(c_{k+1}) = f_k(\partial c_k)$ where $c_k \in C_k$, and indeed δ is a valid boundary map as $\delta \circ \delta(f_k)(c_{k+2}) = f_k\left(\partial \circ \partial(c_{k+2})\right) = f_k(0) = 0$ as f_k is a homomorphism of abelian groups. We can then "take the cohomology" of this cochain complex in a similar manner to before, by defining $H^k = \ker(\delta_k)/\text{im}(\delta_{k-1})$, and, in full analogy, elements of the kernel of δ are called cocyles, and elements of the image of δ are called coboundaries. For example, applying this procedure to singular homology, we get singular cohomology. We will explore cohomology theories further in section 3.

2.1.3 Equivalence of simplicial and singular homology

Even though singular homology might appear to be quite different to simplicial homology, especially in view of the size of the groups appearing in the quotient $H_n(X)$, the two theories always agree (when they are both defined). Let f_i be the homomorphisms

[18] A cochain complex is akin to a chain complex, as it is also a sequence of abelian groups connected by a boundary operator, except that the boundary operator raises degree instead of lowering it.

$C'_n(X,A) \to C_n(X,A)$ defined by sending each n-simplex $\Delta^n \in C'_n(X)$ of X to its corresponding singular n-simplex given by its characteristic map $\sigma\colon \Delta^n \to X$
f_i is a chain map so induces homomorphisms $f_*\colon H'_n(X,A) \to H_n(X,A)$.

Theorem 2.18 The homomorphisms $f_*\colon H'_n(X,A) \to H_n(X,A)$ are isomorphisms.

Proof. Following Hatcher [Hat02], but we will only consider the case when X is finite dimensional. First, suppose $A = \emptyset$.
Write X^k for the k-skeleton of X (X is a simplicial complex). We then have an exact sequence

$$H_{n+1}(X^k, X^{k-1}) \longrightarrow H_n(X^{k-1}) \longrightarrow H_n(X^k) \longrightarrow H_n(X^k, X^{k-1}) \longrightarrow H_{n-1}(X^{k-1})$$

and the same one for simplicial homology, which come from the long exact sequence in relative homology of Proposition 2.14. This allows us to form the following commutative diagram

$$\begin{array}{ccccccccc}
H'_{n+1}(X^k, X^{k-1}) & \longrightarrow & H'_n(X^{k-1}) & \longrightarrow & H'_n(X^k) & \longrightarrow & H'_n(X^k, X^{k-1}) & \longrightarrow & H'_{n-1}(X^{k-1}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_{n+1}(X^k, X^{k-1}) & \longrightarrow & H_n(X^{k-1}) & \longrightarrow & H_n(X^k) & \longrightarrow & H_n(X^k, X^{k-1}) & \longrightarrow & H_{n-1}(X^{k-1})
\end{array}$$

(which commutes because the vertical maps form a chain map). $C'_n(X^k, X^{k-1})$ is 0 for $n \neq k$ and free abelian with basis in bijection with the k-simplices of X when $n = k$, so we can say the same of $H'_n(X^k, X^{k-1})$. For the corresponding singular homology group $H_n(X^k, X^{k-1})$, we are led to consider the map $\coprod_i (\Delta^k_i, \partial\Delta^k_i) \to (X^k, X^{k-1})$ consisting of all characteristic maps $\Delta^k \to X$. This induces isomorphisms of the quotient spaces $\coprod_i \Delta^k_i / \coprod_i \partial\Delta^k_i \cong X^k/X^{k-1}$, hence it induces isomorphisms on the singular homology groups, so that $H_n(X^k, X^{k-1})$ has the same description as $H'_n(X^k, X^{k-1})$, 0 for $n \neq k$ and free abelian with basis in bijection with the characteristic maps of the singular k-simplices of X when $n = k$. This is what the vertical map induces, hence the first and fourth columns in the diagram are isomorphisms. We can also consider the second and fifth columns to be isomorphisms: they obviously are isomorphisms for X^0, and if we assume this fact for X^{k-1} we get the result for X^k, hence allowing us to use induction.
We can now use the following algebraic result:

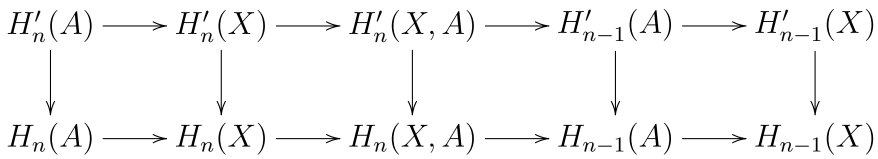
Lemma 2.19 (Five Lemma)
Suppose the following diagram of abelian groups and homomorphisms commutes, where the two rows are exact sequences:

$$\begin{array}{ccccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \varepsilon} \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E'
\end{array}$$

Then, if α, β, δ and ε are all isomorphisms, γ is an isomorphism too.

Proof. Omitted, see [Mac98]. ❖

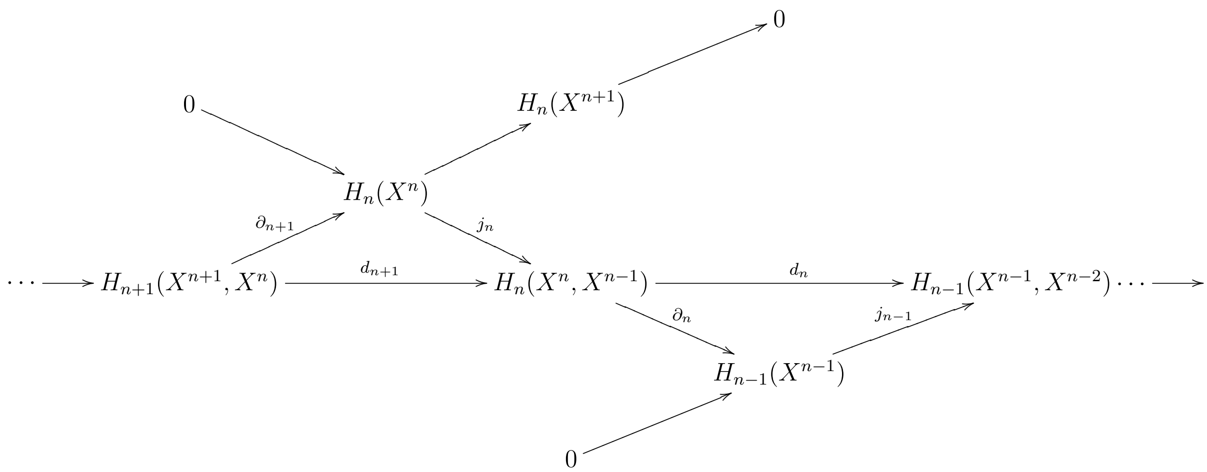
This then gives the required equivalence when $A = \emptyset$. Now, suppose $A \neq \emptyset$; from Proposition 2.14, we have the following commutative diagram:



in which the rows are exact, and the first, second, fourth and fifth columns are isomorphisms. So we can apply the Five Lemma again, to obtain the desired equivalence. ❖

2.1.4 Cellular homology

The proof of Theorem 2.18 hints towards the use of the groups $H_n(X^k, X^{k-1})$. As we then noted, $H_n(X^k, X^{k-1})$ is 0 for $k \neq n$ and is a free abelian group when $n = k$, with basis in bijection with the k-cells of X. Some other facts of importance are noted in [Hat02], for example, that $H_n(X^k) \cong 0$ for $n > k$ and that the inclusion $i\colon X^k \to X$ induces isomorphisms $i_*\colon H_n(X^k) \to H_n(X)$ for all $n < k$. This is really what we would expect, so we will not prove this; a proof can be found in [Hat02]. We can then look at portions of the long exact sequences corresponding to the pairs $(X^{k+1}, X^k), (X^k, X^{k-1}), (X^{k-1}, X^{k-2}), \ldots,$ and weave them together to form the following commutative diagram:



Where $d_i = j_{i-1}\partial_i$ are just the composites coming from weaving the portions together, they allow us to consider the horizontal sequence. Hence the composition $d_k d_{k-1}$ is always

0, so the horizontal sequence is indeed a chain complex

$$\cdots \longrightarrow H_{n+1}(X^{n+1}, X^n) \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2}) \longrightarrow \cdots$$

called the cellular chain complex of X. The homology of this complex is then called the cellular homology of X, denoted $H_\bullet^{CW}$.

Theorem 2.20 $H_n^{CW}(X) \cong H_n(X)$ for all n.

Proof. From the above diagram, we see that $H_n(X) \cong H_n(X^n)/\mathrm{im}(\partial_{n+1})$. From exactness, we know that j_n is injective, therefore $\mathrm{im}(\partial_{n+1}) = \mathrm{im}(j_n\partial_{n+1}) = \mathrm{im}(d_{n+1})$ from commutativity. Similarly, we get an isomorphism $H_n(X^n) \cong \mathrm{im}(j_n) = \ker(\partial_n)$ by exactness. j_{n-1} is also injective, so $\ker(\partial_n) = \ker(d_n)$. Hence we have that $H_n(X^n)/\mathrm{im}(\partial_{n+1}) \cong \ker(d_n)/\mathrm{im}(d_{n+1})$. But we know that the right hand side is the cellular homology, and that $H_n(X) \cong H_n(X^n)/\mathrm{im}(\partial_{n+1})$. Hence the theorem holds. ❖

One trouble with the calculation of homotopy groups was that maps can be very bizarre, for example the fact that the image of a circle in S^2 might be dense, even surjective. We gave an argument to show that we could consider a homotopy of any such map so that it is no longer surjective; the idea extends more generally in the case of CW complexes:

Definition 2.21 A map $f\colon X \to Y$ of CW complexes is called cellular if $f(X^n) \subset Y^n$ for all n.

Theorem 2.22 (Cellular Approximation)
Any map $f\colon X \to Y$ of CW complexes is homotopic to a cellular map. In addition, if A is a subcomplex of X and $f|_A\colon A \to Y$ is cellular, then $f\colon X \to Y$ is homotopic to a cellular map $g\colon X \to Y$ such that $g|_A = f|_A$.

Proof. Omitted, see [Hat02]. ❖

For instance, this allows us to prove more rigorously that $\pi_n(S^k) = 0$ for all $n < k$: any map $S^n \to S^k$ is homotopic to a cellular map, but $S^k \simeq D^0 \sqcup D^k$, hence any such map must be homotopic to the constant map.
The following theorem shows us that the homotopy groups of CW complexes encapsulate all the information we need to know. This is perhaps not very surprising, as CW complexes are built up using spheres.

Theorem 2.23 (Whitehead)
If a map between CW complexes induces isomorphisms on all homotopy groups, then it is in fact a homotopy equivalence.

Proof. See [Hat02]. ❖

A map which induces isomorphisms on all homotopy groups is called a weak homotopy equivalence. The theorem then says that a weak homotopy equivalence of CW complexes is in fact a homotopy equivalence. Notice however that it is not true, in general, that two CW complexes with isomorphic homotopy groups are homotopy equivalent: the isomorphisms must be induced by some map.
We also note the following theorem, which will be useful for many heuristic arguments:

Theorem 2.24 (CW approximation)
For any topological space X there exists a CW complex Y with a weak homotopy equivalence $X \to Y$.

Proof. See [Hat02]. ❖

2.1.5 Axioms for homology

A convenient way to unify all the different homology theories we have so far is to propose some axioms that all satisfy – this motivates the following set of axioms:

Definition 2.25 (Eilenberg-Steenrod axioms)
A (reduced) homology theory assigns to each (nonempty) CW complex X a sequence of abelian groups $(\widetilde{H}_n(X))$ and to each map $f\colon X \to Y$ of CW complexes an induced homomorphism $f_*\colon \widetilde{H}_n(X) \to \widetilde{H}_n(Y)$ for all n, so that $(fg)_* = f_* g_*$, $\mathrm{id}_* = \mathrm{id}$, with maps $\partial_n\colon \widetilde{H}_n(X/A) \to \widetilde{H}_{n-1}(A)$,[19] satisfying the following axioms:
Homotopy equivalence If continuous maps f and g from X to Y are homotopic, then $f_* = g_*$ for all n.
Additivity If $X = \bigvee_i X_i$ then $\widetilde{H}_n(X) = \bigoplus_i \widetilde{H}_n(X_i)$ for all n.
Dimension If P is a point, then $\widetilde{H}_n(P) \cong 0$ for all $n \neq 0$.
Long exact sequence The inclusion $i\colon A \to X$ and the quotient map $q\colon X \to X/A$ induce the long exact sequence

$$\widetilde{H}_{n+1}(X/A) \overset{\partial_{n+1}}{\to} \widetilde{H}_n(A) \overset{i_*}{\to} \widetilde{H}_n(X) \overset{q_*}{\to} \widetilde{H}_n(X/A) \overset{\partial_n}{\to} \cdots \overset{q_*}{\to} \widetilde{H}_0(X/A) \to 0$$

In a similar fashion, we can come up with axioms for a relative, nonreduced homology theory $H_n(X, A)$, we recover the absolute version by setting $H_n(X) = H_n(X, \emptyset)$, and a re-

[19]The maps ∂_n should in fact be "natural", which means that for each map $f\colon (X, A) \to (Y, B)$ that induces a quotient map $\overline{f}\colon X/A \to Y/B$, there is a commutative diagram

$$\begin{array}{ccc} \widetilde{H}_n(X/A) & \xrightarrow{\partial_n} & \widetilde{H}_{n-1}(A) \\ \downarrow{\scriptstyle \overline{f}_*} & & \downarrow{\scriptstyle f_*} \\ \widetilde{H}_n(Y/B) & \xrightarrow{\partial_n} & \widetilde{H}_{n-1}(B) \end{array}$$

In terms of category theory, this just means that the homology $\widetilde{H}$ is a sequence of functors $\widetilde{H}_n$ from the category of CW complexes to the category of abelian groups, with a natural transformation $\partial_n\colon \widetilde{H}_n(X/A) \to \widetilde{H}_{n-1}(A)$ for all n.

duced theory $\widetilde{H}_n(X)$ by setting it equal to the kernel of the natural map $H_n(X) \to H_n(P)$. On the other hand, we can get a unreduced version by setting $H_n(X) = \widetilde{H}_n(X \coprod P)$. Most changes to the axioms are straightforward, but the long exact sequence needs to be split up in two parts: the long exact sequence of relative homology and excision. We then get the following axioms:

Definition 2.26 (Axioms for unreduced homology)
A homology theory assigns to each pair of topological spaces (X, A) a sequence of abelian groups $(H_n(X, A))$ and to each map $f\colon (X, A) \to (Y, B)$ of pairs of topological spaces an induced homomorphism $f_*\colon H_n(X) \to H_n(Y)$ for all n, so that $(fg)_* = f_* g_*$, $\mathrm{id}_* = \mathrm{id}$, with (natural) maps $\partial_n\colon H_n(X, A) \to H_{n-1}(A)$, satisfying the following axioms:
Homotopy equivalence If f and g are homotopic, then $f_* = g_*$ for all n.
Additivity If $X = \coprod_i X_i$ then $H_n(X) = \bigoplus_i H_n(X_i)$ for all n.
Dimension If P is a point, then $H_n(P) \cong 0$ for all $n \neq 0$.
Excision If $U \subset A$ and the closure of U is contained in the interior of A, then the inclusion $i\colon (X \setminus U, A \setminus U) \to (X, A)$ induces an isomorphism in homology $i_*\colon H_n(X \setminus U, A \setminus U) \to H_n(X, A)$.
Long exact sequence The inclusions $i\colon A \to X$ and $j\colon X \to (X, A)$ induce the long exact sequence

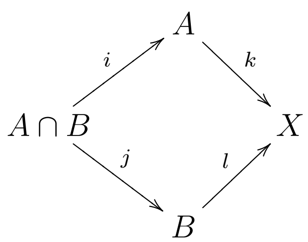
$$H_{n+1}(X, A) \xrightarrow{\partial_{n+1}} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial_n} \cdots \xrightarrow{j_*} H_0(X, A) \to 0$$

Not long after Eilenberg and Steenrod first formulated these axioms, new homology theories were discovered, except that they did not satisfy the dimension axiom: they are called "extraordinary" homology theories, the first two examples were of K-theory and of bordism theory.

2.2 Relation with homotopy

2.2.1 Mayer-Vietoris sequence

The Mayer-Vietoris sequence is a homological analog of the Van Kampen Theorem as it allows the computation of the homology of $A \cup B$ from that of A, B and $A \cap B$.
Suppose A and B are two subspaces of X such that $X = \mathrm{int}(A) \cup \mathrm{int}(B)$ is the union of the interiors of A and B. We then have the following commutative diagram:



Where i, j, k, l are the standard inclusions. This induces the short exact sequence

$$0 \to C_n(A \cap B) \stackrel{(i_*,-j_*)}{\to} C_n(A) \oplus C_n(B) \stackrel{k+l}{\to} C_n(A+B) \to 0$$

where $C_n(A+B)$ is the subgroup of $C_n(X)$ of n-chains that are the sum of chains in A and chains in B.
We then have a short exact sequence of chain complexes:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{n+1}(A \cap B) & \longrightarrow & C_{n+1}(A) \oplus C_{n+1}(B) & \longrightarrow & C_{n+1}(A+B) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_n(A \cap B) & \longrightarrow & C_n(A) \oplus C_n(B) & \longrightarrow & C_n(A+B) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{n-1}(A \cap B) & \longrightarrow & C_{n-1}(A) \oplus C_{n-1}(B) & \longrightarrow & C_{n-1}(A+B) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

This diagram commutes as the horizontal arrows are inclusions, so the boundary operator acts in the same way in all three columns (modulo restriction to A,B or $A \cap B$). The Zigzag lemma (Lemma 2.13) then shows that there exists a long exact sequence of homology:

$$H_{n+1}(X) \stackrel{d_{n+1}}{\to} H_n(A \cap B) \stackrel{(i_*,-j_*)}{\to} H_n(A) \oplus H_n(B) \stackrel{k_*+l_*}{\to} H_n(X) \stackrel{d_n}{\to} \cdots \stackrel{k_*+l_*}{\to} H_0(X) \to 0$$

Here $H_n(A+B)$ has turned into $H_n(X)$ as the inclusion $e\colon C_n(A+B) \to C_n(X)$ induces an isomorphism in homology $e_*\colon H_n(A+B) \to H_n(X)$ by Remark 2.17.
Then any $\sigma \in H_n(X)$ can be written as $\eta+\mu \in C_n(A+B)$, and we have that $\partial(\eta+\mu) = 0$ so $\partial\eta = -\partial\mu$, which need not be zero as η and μ are not necessarily cycles, even though ω is. This allows us to write the "connecting homomorphism" $d\sigma = \partial\eta \in H_{n-1}(A \cap B)$.
It is also possible to consider the Mayer-Vietoris sequence for reduced homology, which is

obtained by consideration of the same diagram except for the bottom row:[20]

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_1(A\cap B) & \longrightarrow & C_1(A)\oplus C_1(B) & \longrightarrow & C_1(A+B) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_0(A\cap B) & \longrightarrow & C_0(A)\oplus C_0(B) & \longrightarrow & C_0(A+B) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Again, the Zigzag Lemma shows that there is an analog long exact sequence in reduced homology:

$$\widetilde{H}_{n+1}(X) \overset{d_{n+1}}{\to} \widetilde{H}_n(A\cap B) \overset{(i_*,-j_*)}{\to} \widetilde{H}_n(A)\oplus\widetilde{H}_n(B) \overset{k_*+l_*}{\to} \widetilde{H}_n(X) \overset{d_n}{\to} \cdots \overset{k_*+l_*}{\to} \widetilde{H}_0(X)\to 0$$

The Mayer-Vietoris sequence gives a way to compute the homology of the Klein bottle: consider the Klein bottle K as the union of two Möbius strips A, B along their boundaries.[21] A, B and $A\cap B$ are all homotopy equivalent to circles, so we have the exact sequence

$$0\to\widetilde{H}_2(K)\to\widetilde{H}_1(A\cap B)\to\widetilde{H}_1(A)\oplus\widetilde{H}_1(B)\to\widetilde{H}_1(K)\to 0\to\cdots$$

which we can rewrite as

$$0\to\widetilde{H}_2(K)\overset{d}{\to}\mathbb{Z}\overset{(i_*,-j_*)}{\to}\mathbb{Z}^2\overset{k_*+l_*}{\to}\widetilde{H}_1(K)\to 0$$

The map $(i_*,-j_*)\colon \mathbb{Z}\to\mathbb{Z}^2$ sends 1 to $(2,-2)$ as the inclusion i (same for j) induces a map $i_*\colon H_1(A\cap B)\to H_1(A)$, $i_*\colon\mathbb{Z}\to\mathbb{Z}$ that maps 1 to 2 as the boundary of a Möbius strip winds twice around its central circle. $(i_*,-j_*)$ is injective, so exactness at $\mathbb{Z}$ implies that the map $\widetilde{H}_2(K)\overset{d}{\to}\mathbb{Z}$ is 0, hence $\widetilde{H}_2(K)\cong 0$ by exactness at $\widetilde{H}_2(K)$. We can write every element in $\mathbb{Z}^2$ as $a(1,-1)+b(1,0)$ for $a,b\in\mathbb{Z}$, with exactly the elements of the form $a\in 2\mathbb{Z}$, $b=0$ in the image of $(i_*,-j_*)$, hence in the kernel of the map $k_*+l_*\colon\mathbb{Z}^2\to\widetilde{H}_1(K)$ by exactness. So $\widetilde{H}_1(K)\cong\mathbb{Z}^2/\ker(k_*+l_*)\cong\mathbb{Z}\oplus\mathbb{Z}_2$ by the First Isomorphism Theorem. The Mayer-Vietoris sequence also allows easy computation of $\widetilde{H}_n(S^k)$, as follows. Consider

[20]Once again, we should require A, B and $A\cap B$ to be nonempty to avoid having problems with $H_{-1}(\emptyset)$.

[21]In fact, we have to let the two Möbius strips overlap somewhat for the Mayer-Vietoris sequence to exist, as we need K to be covered by the interior of the strips.

S^k as the union of two halves A and B of S^k along the equator S^{k-1}.[22] A and B are both contractible, so we are left with the exact sequence

$$0 \to \widetilde{H}_n(S^k) \xrightarrow{d} \widetilde{H}_{n-1}(S^{k-1}) \to 0$$

Hence d is an isomorphism by exactness. We know that $\widetilde{H_n}(S^0)$ is isomorphic to $\mathbb{Z}$ if $n = 0$, and otherwise is 0. Therefore, by induction, $\widetilde{H}_n(S^n) \cong \mathbb{Z}$ and $\widetilde{H}_n(S^k) = 0$ for $n \neq k$.

2.2.2 Hurewicz Theorem

The fundamental group $\pi_1(X)$ and the first homology group $H_1(X)$ are closely related by the fact that we can consider a map $S^1 \to X$ both as a loop or as a singular 1-simplex. We might wonder if this consideration gives us an isomorphism between H_1 and π_1, but we know that this is not the case as H_1 is necessarily abelian whereas π_1 isn't. In fact, this is the only obstruction:

Theorem 2.27 Suppose X is path connected. Considering loops as singular 1-simplices, we get a homomorphism $h\colon \pi_1(X) \to H_1(X)$ which is surjective with kernel $\ker(h) = [\pi_1(X), \pi_1(X)]$. Hence $H_1(X)$ is the abelianisation of $\pi_1(X)$.

Proof. Omitted, see [Hat02]. ❖

In a similar way, we also have a canonical homomorphism from π_n to H_n by considering the image of the n-sphere as a singular n-simplex. This map is not always an isomorphism, for example $\pi_3(S^2) \cong \mathbb{Z}$ but $H_3(S^2) \cong 0$. It is however an isomorphism under certain special conditions:

Theorem 2.28 (Hurewicz)
Let X be an $(n-1)$-connected space[23] with $n \geqslant 2$.
Then $\widetilde{H}_k(X) = 0$ for $0 \leqslant k < n$ and $\pi_n(X) \cong H_n(X)$

Proof. Omitted. See [Hat02]. ❖

In general, this homomorphism $h\colon \pi_n(X) \to H_n(X)$ is called the Hurewicz homomorphism.
In the case of the n-sphere, we know that the first non-zero homotopy group is $\pi_n(S^n)$ (using Theorem 2.22). Therefore, this gives us a proof that $\pi_n(S^n) \cong H_n(S^n) \cong \mathbb{Z}$.

[22] Again, we must allow the halves to overlap somewhat, so that S^k is covered by the interiors of A and B

[23] X is $(n-1)$-connected means that X is path connected and that $\pi_i(X) = 0$ for $i < n$.

3 Cohomology

3.1 Manifolds and differential forms

Manifolds

A motivation for the study of manifolds is the following theorem:

Theorem 3.1 (Classification of surfaces)
Every connected closed surface is homeomorphic either to the sphere, the sphere with n handles attached, or the sphere with n discs replaced by Möbius strips. In addition, no two of these surfaces are homeomorphic.

Proof. Omitted, see [Arm83]. ❖

The precise meaning of the above theorem is as follows: a surface is a Hausdorff topological space in which every point has a neighbourhood homeomorphic to the plane $\mathbb{R}^2$. Notice that such a surface cannot have a boundary; sometimes a boundary is allowed, and for a surface with boundary, we must consider the upper half plane $H = \{(x, y) \in \mathbb{R}^2 : y \geqslant 0\}$ in addition to $\mathbb{R}^2$. A surface is then closed if it is compact and without boundary. This has the following obvious generalisation:

Definition 3.2 A (topological) manifold M of dimension $n \geqslant 0$ is a Hausdorff, second countable topological space with an open cover $\{U_i\}$ such that U_i is homeomorphic to $\mathbb{R}^n$ for all i.

Remark 3.3 The assumption that M be second countable is sometimes dropped. However, every Hausdorff second countable manifold is metrisable; this is a corollary of Urysohn's Metrisation Theorem, which states that every second countable Hausdorff regular space is metrisable.

Remark 3.4 In particular, with our definition, every manifold admits partitions of unity, which are a very useful tool. For example, given an open cover $\mathcal{U} = \{U_i\}$, a partition of unity subordinate to this open cover is a set of continuous functions $\{F_j\}$ with sum 1, only finitely many nonzero, such that, for all j there is an i with $\operatorname{supp}(F_j) \subset U_i$. Also note that this definition is again for manifolds without boundary, if we allow for a boundary we must also consider $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \geqslant 0\}$. It then follows that the interior of a manifold with boundary is a manifold without boundary of the same dimension, and the boundary of a manifold with boundary is a manifold without boundary of dimension one less. We will be taking "manifold" to mean manifold without boundary, unless noted otherwise.

We would also like to have some additional structure on our manifold, some geometry, some differentiable structure. To do this, we had better require that, instead of simple homeomorphisms, we consider diffeomorphisms. But, whereas patching together continuous functions always gives rise to continuous functions, it is not true that patching together differentiable functions gives a differentiable function. Hence we had better require some compatibility between the different diffeomorphisms. To do this, we first consider the "charts" $\varphi_i \colon U_i \to \mathbb{R}^n$, which are the homeomorphisms in the definition. We call $\{\varphi_i\}$ an atlas of charts on the manifold. What we then want to look at are the "transition maps" $t_{ij} = \varphi_i \circ \varphi_j^{-1}$ defined on the intersection $U_i \cap U_j$, so $t_{ij} \colon \mathbb{R}^n \to \mathbb{R}^n$. This allows us to formulate the following definition:

Definition 3.5 A differentiable manifold of class C^k is a topological space equipped with an atlas of charts $\{\varphi_i\}$ such that the transitions maps t_{ij} are differentiable of class C^k.[24]

We will mostly be considering smooth manifolds, which are differentiable manifolds of class C^∞. This avoids any technical considerations of differentiability.

Bundles

A convenient setting for talking about derivatives is given by consideration of tangent spaces, for example tangent planes to a surface. There are many equivalent ways of defining such a tangent space for a differentiable manifold; for example as the set of tangent vectors to curves. But we can also look at a tangent vector as something which differentiates, by giving a "directional derivative" of smooth maps $M \to \mathbb{R}$ (this set of smooth maps is denoted $C^\infty(M)$).

Definition 3.6 A derivation is a linear map $D \colon C^\infty(M) \to \mathbb{R}$ satisfying the "Leibniz law" $D(fg) = (Df)g + f(Dg)$.
The tangent space T_xM to M at x is the vector space of all derivations $D \colon C_x^\infty(M) \to \mathbb{R}$, where $C_x^\infty(M)$ is the set of smooth functions defined in a neighbourhood of x.
The tangent bundle TM is the disjoint union $\coprod_{x \in M} T_xM$.

Remark 3.7 There are some facts here that have gone without justification: it is indeed the case that T_xM is a vector space, in fact it has the same dimension as the manifold. A proof can be found in [Spi99]. It is also the case that TM has a natural topology which makes it into a manifold of dimension twice that of M, see [Mor01].

Given a chart φ, we write $\frac{\partial}{\partial x_i}$ for the derivation corresponding to the partial derivative along the i-th coordinate of the chart. Hence for a function $f \colon M \to \mathbb{R}$ and a tangent vector $X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$, we can write $X(f) = \sum_{i=1}^n a_i \frac{\partial f}{\partial x_i}$.
We can also form the cotangent space $T_x^*M = (T_xM)^*$ as a dual space, and then the cotangent bundle $T^*M = \coprod_{x \in M} T_x^*M$. Locally, we write the dual covector to $\frac{\partial}{\partial x_i}$ by dx_i, so that $dx_i \left(\frac{\partial}{\partial x_j} \right) = \delta_i^j$ which is 1 if $i = j$ and 0 otherwise.

[24]We do not need to require them to be diffeomorphisms as $t_{ji} = (t_{ij})^{-1}$. In fact, we could notice that different atlases can be "equivalent": we say two C^k atlases $\{\varphi_i\}$ and $\{\psi_i\}$ are equivalent if $\{\varphi_i, \psi_j\}$ is also a C^k atlas. This does define an equivalence relation, and we call an equivalence class a differentiable structure on our manifold.

Notice that for small enough open sets U (for example, if U is contractible), the tangent bundle TU is diffeomorphic to $U \times \mathbb{R}^n$. This important property is that of a fiber bundle:

Definition 3.8 A fiber bundle $\xi = (\pi, F, E, M)$ is given by a continuous, surjective map $\pi\colon E \to M$ such that each point $x \in M$ has a neighbourhood U such that $\pi^{-1}(U)$ is homeomorphic to $U \times F$. This projection has to satisfy the commutative diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{f} & U \times F \\ \downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle p_1} & \\ U & & \end{array}$$

where p_1 is the projection onto the first factor and f is the required homeomorphism. We call F the fiber of the bundle, E the total space and M the base space.

It is important to note that although a fiber bundle is locally a product, it is not necessarily globally a product. For example, the tangent bundle to the sphere, TS^2, cannot be $S^2 \times R^2$, as this would contradict Theorem 4.5 which we prove later on. For this reason, we say a fiber bundle which is globally a product is trivial, and so general fiber bundles are only locally trivial.

But TM (and T^*M) come with an additional structure, because the fiber F is a vector space. This gives the idea of a vector bundle:

Definition 3.9 A (real) vector bundle of rank r is a fiber bundle $\xi = (\pi, F, E, M)$ with fiber $F = \mathbb{R}^r$, such that the homeomorphisms f in the definition of a fiber bundle satisfy the additional property that for all $p \in U$ and $v \in F$, the map $v \mapsto f(x, v)$ is a linear isomorphism of vector spaces.

We can also consider maps between fiber bundles $\xi_1 = (\pi_1, F_1, E_1, M_1)$ and $\xi_2 = (\pi_2, F_2, E_2, M_2)$, which are just maps f, g that make the following diagram commute:

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow{g} & M_2 \end{array}$$

To be a vector bundle morphism, we have to add the condition that the induced maps $\pi^{-1}(x) \to \pi^{-1}(g(x))$ are linear maps of vector spaces.

In particular, given a fiber bundle $\xi = (\pi, F, E, M)$ and a map $f\colon N \to M$, we can form the pullback bundle $f^*(\xi) = (\varpi, F', E', N)$ by $E' = \{(x, e) \in N \times E : f(x) = \pi(e)\}$, $\varpi(x, e) = x$. Then $F' = F$ as $\varpi^{-1}(x) = \pi^{-1} f(x)$.

Now, given $\pi\colon TM \to M$, we can construct a vector field X by choosing, for each point in M, a vector in the fiber T_xM. For a general fiber bundle, this idea is that of a section:

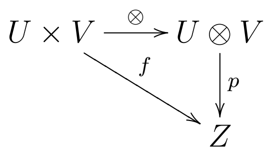
Definition 3.10 Let $\xi = (\pi, F, E, M)$ be a fiber bundle. A section of ξ is a map $s\colon M \to E$ such that $\pi(s(p)) = p$ for all $p \in M$. The set of all sections of ξ is denoted

$\Gamma(E)$.

Then a (smooth) vector field is simply a (smooth) section of the tangent bundle. We would like to generalise cotangent vectors to functions that take more than one tangent vector and give out a real number, linear in each variable. This precisely corresponds to taking tensor products of cotangent vectors.

Tensors

Given two vector spaces U and V over a field K, we can form the vector space $U \otimes_K V$ as follows:
Consider the vector space W which has as basis elements of the form (u, v) with $u \in U$ and $v \in V$.
The vector space $U \otimes_K V$ (from now on $U \otimes V$) is then the quotient of W under the following identifications:
$(u_1 + u_2, v) = (u_1, v) + (u_2, v)$
$(u, v_1 + v_2) = (u, v_1) + (u, v_2)$
$(\lambda u, v) = (u, \lambda v) = \lambda(u, v)$ for $\lambda \in K$
We then write $u \otimes v$ for the image of (u, v) under the identification.[25]
The most important feature of the tensor product is the following "universal property":
Given any bilinear map $f\colon U \times V \to Z$, there exists a unique linear map $p\colon U \otimes V \to Z$ that makes the following diagram commute:



This is because a bilinear map $f\colon U \times V \to Z$ can just be seen as a map $\widetilde{f}\colon U \otimes V \to Z$ by $\widetilde{f}(u \otimes v) = f(u, v)$, and the fact that f is bilinear implies precisely that the map $\widetilde{f}$ is linear by the quotients we took to get $U \otimes V$ out of W. As all three quotients are necessary, we can see that we have constructed the most general space we could that has these properties, so this gives an idea of why this universal property holds.
We can therefore see the tensor product as being the "most general bilinear operation", as any bilinear map from $U \times V$ must factor through $U \otimes V$. Indeed, the above commutative diagram implies that $\mathrm{L}^2(U \times V, Z) \cong \mathrm{L}(U \otimes V, Z)$.
Given a vector space V, we can consider tensors on V:

Definition 3.11 $T^m_n(V) = L^{m+n}(\underbrace{V^*, \dots, V^*}_{m}, \underbrace{V, \dots, V}_{n}; \mathbb{R}) \cong \underbrace{V \otimes \cdots \otimes V}_{m} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{n}$.[26]

[25] In fact, the same construction works for modules over a commutative ring without any changes. In particular, this works for abelian groups.
[26] That is, the set of $m + n$-linear maps that take m vectors in V^*, n vectors in V, and give a real number. The equality on the right follows from the previous discussion of tensor products and their

Elements of T^m_n are called tensors of type (m, n).

Here we will be mainly interested in the case $V = T_xM$. We can then define the vector bundle of tensors of type (m, n) as $T^m_n(M) = \coprod_{x\in M} T^m_n(T_xM)$. Tensors fields are then smooth sections of $T^m_n(M)$: $\mathcal{T}^m_n = \Gamma(T^m_n)$, vector fields are elements of $\mathcal{T}^1_0$, and covector fields elements of $\mathcal{T}^0_1$.
We can construct the tensor algebra as the space $T(M) = \bigoplus_{k=0}^{\infty} T^0_k(M)$, which is the algebra of tensors of type $(0, k)$ for all $k \in \mathbb{N}$. The product here is given by the tensor product of tensors: $T\otimes S(v_1, \ldots, v_m, v_{m+1}, \ldots, v_{m+n}) = T(v_1, \ldots, v_m)\cdot T(v_{m+1}, \ldots, v_{m+n})$.
There is a particular important construction we can make: the exterior algebra $\Lambda(M) = T(M)/I$, where I is the ideal generated by all elements of the form $x \otimes x$, which gives us antisymmetric tensors.[27]
The product in the exterior algebra $\Lambda(M)$ is given by the wedge product $x \wedge y = [x \otimes y]$ where $[x \otimes y]$ is the representative element of $x \otimes y$ under the quotient by I. [28]
$\Lambda(M)$ is a graded algebra $\Lambda^0(M)\oplus\Lambda^1(M)\oplus\cdots\oplus\Lambda^d(M)$[29] where d is the dimension of M.
We then have the same structure on $\Omega(M) = \Gamma(\Lambda(M)) = \Omega^0(M)\oplus\Omega^1(M)\oplus\cdots\oplus\Omega^d(M)$, and elements of $\Omega^k(M)$ are called differential k-forms.
Hence differential k-forms are, at each point $x \in M$, an alternating k-linear map, which varies smoothly as x is moved. For example, differential one-forms on a 3 dimensional manifold would look locally like $\omega = f_1dx_1 + f_2dx_2 + f_3dx_3$, where the f_i are smooth functions on M and the dx_i form the standard (local) coordinate basis of the cotangent space T_xM, which are dual to the vectors $\frac{\partial}{\partial x_i}$ induced from a chart. Notice that Λ^n and hence Ω^n are 0 for $n > d$, as in a local basis we would have to have repeated factors, but $dx_i \wedge dx_i$ is 0 as it has to be antisymmetric.

Differential forms

Given a map $f\colon M \to N$, we can look at how this affects tangent vectors and differential forms.
Given a tangent vector $X \in T_pM$, f should transform X according to its first-order behaviour. More precisely, f induces a natural map $f_*\colon T_pM \to T_{f(p)}N$ called the pushforward of f by $f_*(X_p)(g) = X_p(g(f))$. This shows that f_* acts on tangent vectors by "scaling" by its differential, as in the usual case of $\mathbb{R}^n$.
Notice that we cannot pushforward vector fields: f is not necessarily injective, for

relation to multilinear maps.
[27]As quotienting out by the relation $x\otimes x = 0$ implies that $(x+y)\otimes(x+y) = 0$, which expanded gives $x\otimes x + y\otimes y + x\otimes y + y\otimes x = x\otimes y + y\otimes x = 0$.
[28]This is equivalent to the following definition, given in [Spi99]:
Let $\mathrm{Alt}(T(x_1, \ldots, x_n)) = \frac{1}{n!}\sum_{\sigma\in S_n} \mathrm{sign}(\sigma)T(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$, then $x\wedge y = \frac{(k+l)!}{k!l!}\mathrm{Alt}(x\otimes y)$ where x and y are k and l-differential forms, respectively. The coefficient $1/n!$ guarantees that $\mathrm{Alt}(\mathrm{Alt}(T)) = \mathrm{Alt}(T)$ as if T is already alternating, the sum is just the addition of $n!$ copies of T.
[29]This follows from the direct sum construction of $T(M)$. This just means that the wedge product of $x \in \Lambda^k(M)$ and $y \in \Lambda^l$ satisfies $x\wedge y \in \Lambda^{k+l}(M)$.

example, which poses a problem for assigning tangent vectors if $f(x) = f(y)$. But inverse images are often better behaved, and this indeed allows us to define the pullback of differential forms: suppose $\omega \in \Omega^k(M)$, then we define the pullback of ω by $f^*(\omega)(v_1, \ldots, v_k) = \omega(f_*(v_1), \ldots, f_*(v_k))$.
This knowledge about how f acts on tangent vectors allows us to formulate many convenient properties such a map can possess. We say that a map f is an immersion if f_* is everywhere injective, and f is a submersion if f_* is everywhere surjective.
The basic examples of such maps are the standard immersion $i\colon R^n \to R^{n+m}$, $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0, \ldots, 0)$ and the standard submersion $s\colon R^{n+m} \to R^n$, $(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) \mapsto (x_1, \ldots, x_n)$. A related concept is that of regular values: for a function $f\colon M \to N$, a point $x \in M$ is a regular point if $f_*\colon T_xM \to T_{f(x)}N$ is surjective, and a point $y \in N$ is called a regular value if all points in $f^{-1}(y)$ are regular points. The interest of such functions lies in the following:

Theorem 3.12 (Preimage Theorem)
Suppose $f\colon M \to N$ is a smooth map, and $y \in Y$ a regular value of f. Then $X = f^{-1}(y)$ is a submanifold of M, with $\dim(X) = \dim(M) - \dim(N)$.

Proof. Omitted, see [Lee00]. ❖

This theorem allows us to check easily if many spaces are manifolds; for example, the n-sphere can be described as $f^{-1}(0)$ with $f(x_0, \ldots, x_n) = 1 - \sum_{i=0}^n x_i^2$, and this does satisfy the conditions of the theorem.

If ω is a differential 0-form (that is, a smooth function f), we have the usual notion of derivative: $d\omega = \sum_i \frac{\partial f}{\partial x_i} dx_i$ which corresponds to the usual gradient of a function. It is possible to generalise this to differential k-forms in the following way: Let $\omega = f dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, then $d\omega = \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. The operator d is defined with this and the fact that $d(\omega + \psi) = d(\omega) + d(\psi)$ as every differential k-form must be the sum of such differential forms with only one component.
To make notation easier, we can use a multi-index notation: given a multi-index $I = (i_1, \ldots, i_k)$, write $f dx_I$ for $f dx_{i_1} \wedge \cdots \wedge dx_{i_k}$.

Theorem 3.13 For all differential forms ω, $d(d\omega) = 0$.

Proof. Again, suppose ω is of the form $\omega = f dx_I$. Then $d\omega = \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j \wedge dx_I$, and

$$d(d\omega) = \sum_{l=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_j \partial x_l} dx_l \wedge dx_j \wedge dx_I$$

but the mixed second partial derivatives are equal, so we can switch round j and l, but as $dx_j \wedge dx_l = -dx_l \wedge dx_j$, we have that $d(d\omega) = -d(d\omega)$ so $d(d\omega) = 0$. As every differential k-form is a sum of such differential k-forms and as the exterior derivative is linear, we have that for all ω, $d(d\omega) = 0$. ❖

Theorem 3.14 If ω is a differential k-form, then $d(\omega \wedge \psi) = d\omega \wedge \psi + (-1)^k \omega \wedge d\psi$

Proof. Consider $\omega = f dx_I$ and $\psi = g dx_J$. We then have:

$$\begin{aligned} d(\omega \wedge \psi) &= d(fg dx_I \wedge dx_J) \\ &= d(fg) \wedge dx_I \wedge dx_J \\ &= ((df)g + f(dg)) \wedge dx_I \wedge dx_J \\ &= dfg \wedge dx_I \wedge dx_J + fdg \wedge dx_I \wedge dx_J \\ &= df \wedge dx_I \wedge g dx_J + (-1)^k f dx_I \wedge dg \wedge dx_J \\ &= d\omega \wedge \psi + (-1)^k \omega \wedge d\psi \end{aligned}$$

❖

This definition of exterior derivative, however, has the disadvantage of depending on a choice of coordinates dx_i, but is in fact uniquely determined by some of its properties we have just shown:

Proposition 3.15 Suppose $\delta \colon \Omega^k(M) \to \Omega^{k+1}(M)$ has the following properties:

$$\delta(\omega + \eta) = \delta(\omega) + \delta(\eta) \tag{1}$$

$$\delta(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\deg(\omega)} \omega \wedge d\eta \tag{2}$$

$$\delta(\delta(\omega)) = 0 \tag{3}$$

$$\delta(f) = df \quad \text{for } f \in C^\infty(M) \tag{4}$$

Then $\delta = d$

Proof. Omitted, see [Spi99]. ❖

We might also want to know how the exterior derivative behaves with respect to pullbacks:

Theorem 3.16 Pullbacks commute with exterior derivatives. That is, for $f \colon M \to N$ and a differential form ω, $f^*(d\omega) = d(f^*(\omega))$.

Proof. Omitted, see [Spi99]. ❖

3.2 De Rham cohomology

De Rham cohomology is defined as the cohomology of the de Rham complex, which is

$$0 \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-1}(M) \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0$$

This is indeed a complex as we have seen that $d^2 = 0$. Taking the cohomology of this complex corresponds to taking the quotient groups $H^n_{dR}(M) = \ker(d)/\mathrm{im}(d)$ where it is understood that $\ker(d)$ and $\mathrm{im}(d)$ are both subgroups of $\Omega^k(M)$. Differential forms ω with

$d\omega = 0$ are called closed (or cocyles) and differential forms ω with a differential form ψ such that $d\psi = \omega$ are called exact (or coboundaries).[30]
Given a map $f\colon M \to N$, we get maps $f^*\colon \Omega^k(N) \to \Omega^k(M)$ for all k, the pullbacks. But $f^*(d\omega) = df^*(\omega)$ by Theorem 3.16, so pullbacks bring cycles to cycles and boundaries to boundaries, so they induce homomorphisms in cohomology, also denoted f^*.[31]
As an example, consider the polar coordinate system (r, θ) on $M = \mathbb{R}^2 \setminus 0$. We can write locally $\theta = \arctan(y/x)$, and then $d\theta$ is a well-defined 1-form on M, $d\theta = \frac{-y dx}{x^2+y^2} + \frac{x dy}{x^2+y^2}$, but θ is not a valid 0-form as it is not continuous (or it is multivalued). Then $d(d\theta) = 0$ but $d\theta$ is not exact (despite the rather confusing notation $d\theta$), so this gives us a nontrivial element $[d\theta] \in H^1(M)$, and in fact it "generates" it, so that $H^1(M) \cong \mathbb{R}$, and we have closed differential forms which are not exact.
We can compare this situation to that of vector fields on $\mathbb{R}^3$; we know that if a vector field V is of the form ∇f for some scalar field f, then $\nabla \times V = \nabla \times \nabla f = 0$. If we are considering a vector field defined on a simply connected domain, we know that this statement has a converse: if $\nabla \times V = 0$ then there is a scalar field f such that $V = \nabla f$. We can rephrase this in the language of differential forms as follows: write $V = (f_x, f_y, f_z)$ as a differential form $\omega = f_x dx + f_y dy + f_z dz$, we have that $d\omega = \left(\frac{\partial f_y}{\partial z} - \frac{\partial f_z}{\partial y}\right) dy \wedge dz + \left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right) dz \wedge dx + \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) dx \wedge dy$ which we then associate with the vector field $w = \left(\frac{\partial f_y}{\partial z} - \frac{\partial f_z}{\partial y}, \frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}, \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) = \nabla \times v$.[32]
Similarly, given a scalar field f considered as a 0-form, we see that df can be associated with ∇f as we saw in the definition of the exterior derivative. Hence we have shown that $H^1(S) = 0$ for a simply connected domain $S \subset \mathbb{R}^3$. The same argument can also be applied to $\nabla \cdot \nabla \times V = 0$, because we know that any vector field V with $\nabla \cdot V = 0$ is of the form $\nabla \times W$ for some vector field W, again on a simply connected domain S; this shows that $H^2(S) = 0$. This generalises to the following fact:

Lemma 3.17 (Poincaré Lemma)
$H^k(\mathbb{R}^n) = 0$ for all $k \geqslant 1$.

Proof. Following [Mon06], we are first going to consider $\pi\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $i\colon \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}$ for $n \geqslant 1$. We know that $\pi \circ i$ is just the identity, so $i^* \circ \pi^*$ is an isomorphism in cohomology. We want to show that $\pi_* \circ i_*$ is (co)chain homotopic to the identity. Now, every differential form on $\mathbb{R}^n \times \mathbb{R}$ is a combination of forms $f(x,t)\pi^*(\omega)$ and $f(x,t)\pi^*(\omega) \wedge dt$, where ω is a differential form on $\mathbb{R}^n$. We define our chain homo-

[30]We can also consider compactly supported cohomology, which arises in the exact same way but considering only compactly supported differential forms ω, so that $\mathrm{supp}(\omega)$ is contained in some compact set, and we then write $H^k_c(M)$ for the k-th compactly supported cohomology group.
[31]As in the case of singular homology, this shows that de Rham cohomology is a functor, here from the category of smooth manifolds to the category of abelian groups; it is easy to see that id_* is the identity on cohomology, and that $(f \circ g)^* = g^* \circ f^*$ (this is the other way round that in the case of homology, which shows that cohomology is a contravariant functor).
[32]To make this association more precise, we would need to develop the idea of Hodge duality, which we will not do here.

topies $K_m\colon H^m(\mathbb{R}^{n+1}) \to H^{m-1}(\mathbb{R}^{n+1})$ as 0 on differential forms of the first type and $\pi^*(\omega)\int_0^t f(x,u)du$ on differential forms of the second type (and extending linearly for differential forms which are linear combinations of the two types).
If a differential form is of the first type, we have that $(\mathrm{id} - \pi^* \circ i^*)(f(x,t)\pi^*(\omega)) = f(x,t)\pi^*(\omega) - f(x,0)\pi^*(\omega)$, $dK = 0$ and

$$\begin{aligned}
Kd\left(f(x,t)\pi^*(\omega)\right) &= K\left(\left(\sum_{i=1}^n \frac{\partial f}{\partial x_i}dx_i + \frac{\partial f}{\partial t}dt\right)\wedge \pi^*(\omega) + f(x,t)d\pi^*(\omega)\right)\\
&= K\left(\frac{\partial f}{\partial x_i}\pi^*(dx_i\wedge\omega) + \frac{\partial f}{\partial t}dt\wedge\pi^*(\omega) + f(x,t)\pi^*(d\omega)\right)\\
&= 0 + K\left((-1)^{\deg(\omega)}\frac{\partial f}{\partial t}\pi^*(\omega)\wedge dt\right) + 0\\
&= (-1)^{\deg(\omega)}\pi^*(\omega)\int_0^t \frac{\partial f}{\partial u}du\\
&= (-1)^{\deg(\omega)}\left(f(x,t)\pi^*(\omega) - f(x,0)\pi^*(\omega)\right)
\end{aligned}$$

which is what we wanted, up to sign, but the signs don't matter because $\pi^* \circ i^*$ is still going to take cycles to boundaries.
If a differential form is of the second type, we have that $(\mathrm{id} - \pi^* \circ i^*)(f(x,t)\pi^*(\omega)\wedge dt) = f(x,t)\pi^*(\omega)\wedge dt$, and

$$\begin{aligned}
dK\left(f(x,t)\pi^*(\omega)\wedge dt\right) &= d\left(\left(\int_0^t f(x,u)du\right)\pi^*(\omega)\right)\\
&= \left(\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\int_0^t f(x,u)du\right)dx_i + \frac{\partial}{\partial t}\left(\int_0^t f(x,u)du\right)dt\right)\wedge\pi^*(\omega)\\
&+ \left(\int_0^t f(x,u)du\right)d\pi^*(\omega)\\
&= \left(\sum_{i=1}^n \pi^*(dx_i\wedge\omega)\int_0^t \frac{\partial f}{\partial x_i}du + (-1)^{\deg(\omega)}f(x,t)\pi^*(\omega)\wedge dt\right)\\
&+ \left(\int_0^t f(x,u)du\right)d\pi^*(\omega)
\end{aligned}$$

$$\begin{aligned} Kd\left(f(x,t)\pi^*(\omega)\wedge dt\right) &= K\left(\sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i \wedge \pi^*(\omega)\wedge dt + f(x,t)d\left(\pi^*(\omega)\wedge dt\right)\right) \\ &= K\left(\sum_{i=1}^n \frac{\partial f}{\partial x_i}\pi^*(dx_i\wedge\omega)\wedge dt + f(x,t)d\pi^*(\omega)\wedge dt\right) \\ &= \sum_{i=1}^n \pi^*(dx_i\wedge\omega)\int_0^t \frac{\partial f}{\partial x_i}du + \left(\int_0^t f(x,u)du\right)d\pi^*(\omega) \end{aligned}$$

therefore $dK - Kd = (-1)^{\deg(\omega)} f(x,t)\pi^*(\omega)\wedge dt$, which is what we wanted (up to sign). We then have a chain homotopy which proves that $H^k(\mathbb{R}^n\times\mathbb{R})\cong H^k(\mathbb{R}^n)$. By induction, and because $H^k(\mathbb{R})\cong\mathbb{R}$ if $k=0$ and 0 otherwise, we see that $H^k(\mathbb{R}^n)=0$ for all $k\geqslant 1$, and $H^0(\mathbb{R}^n)\cong\mathbb{R}$. ❖

Corollary 3.18 De Rham cohomology is homotopy invariant. That is, homotopic maps induce the same homomorphisms in cohomology.

Proof. Using the same proof as for the Poincaré Lemma, we can show that $H^k(M\times\mathbb{R})\cong H^k(M)$, given by inverse isomorphisms π^* and i^*. Therefore, if we are given a (smooth) homotopy $F\colon M\times\mathbb{R}\to N$ with $F(x,0)=f(x)$, $F(x,1)=g(x)$, we see that $f^* = i_0^*\circ F^*$ and $g^*=i_1^*\circ F^*$, where $i_0(x)=(x,0)$ and $i_1(x)=(x,1)$; we now know that i_0^* and i_1^* are both inverses to π^*, so they're equal. Therefore $f^*=g^*$. ❖

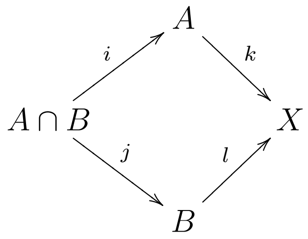
Remark 3.19 In particular, as noted in Corollary 2.9, this shows that homotopy equivalent spaces have isomorphic de Rham cohomology groups.

We can also notice that cohomology has some extra structure compared to homology, as we have a product of chains that turns the cohomology into a ring $H^\bullet$, in fact a ($\mathbb{Z}$) graded ring, by $H^\bullet(M)=\bigoplus_{i=0}^n H^i(M)$ with addition $\Omega^k(M)\times\Omega^k(M)\to\Omega^k(M)$ and (wedge) product $\Omega^k(M)\times\Omega^l(M)\to\Omega^{k+l}(M)$ that pass to the quotient (for addition this is clear, for multiplication it follows from the Leibniz rule of the exterior derivative). An illuminating explanation from [Hat02] for this extra structure is given by the fact that we can see multiplication as a composite $H^k(M)\times H^l(M)\to H^{k+l}(M\times M)\to H^{k+l}(M)$[33]. And for cohomology the second map can be seen to be induced by the map $\Delta\colon M\to M\times M$, $\Delta\colon x\mapsto(x,x)$, whereas for homology, with $H_{k+l}(M\times M)\to H_{k+l}(M)$, the product would be induced by a map $M\times M\to M$, are there is no natural choice of such map (at least, not when M doesn't have any additional structure).

We also note that we can similarly characterise cohomology theories by axioms, these axioms are just the categorical duals of those given for homology (which really means we just need to reverse arrows, but this also changes a few subtle facts, for example direct sums are turned into direct products). From this, we automatically obtain many results

[33]See Section 4.7 (Kunneth Formula) for more details.

we derived early for homology theories, such as a Mayer-Vietoris sequence:
Recalling the diagram



we have the long exact sequence in de Rham cohomology

$$\cdots \to H^n(X) \stackrel{k^*+l^*}{\to} H^n(A) \oplus H^n(B) \stackrel{(i^*,-j^*)}{\to} H^n(A \cap B) \stackrel{\delta}{\to} H^{n+1}(X) \to \cdots$$

For compactly supported cohomology, the pullback of a compactly supported differential form is not necessarily compactly supported, but we also get covariant maps i_*, j_*, k_* and l_* obtained by extending the compactly supported differential forms by 0, so that we get the long exact sequence

$$\cdots \to H^n(A \cap B) \stackrel{(i_*,-j_*)}{\to} H^n(A) \oplus H^n(B) \stackrel{k_*+l_*}{\to} H^n(X) \stackrel{\delta'}{\to} H^{n+1}(X) \to \cdots$$

3.3 Stokes's Theorem and de Rham's Theorem

Consider the Kelvin-Stokes Theorem:

Theorem 3.20 (Kelvin-Stokes)
In $\mathbb{R}^3$, let v be a (smooth) vector field and S an orientable surface. Then

$$\int_S (\nabla \times v) \cdot ds = \oint_{\partial S} v \cdot dr$$

If, instead, we consider the surface S as a submanifold of $\mathbb{R}^3$, we saw above that we can write $v = (f_x, f_y, f_z)$ as a differential form $\omega = f_x dx + f_y dy + f_z dz$, and then we can associate $d\omega$ with $w = \nabla \times v$. Then Kelvin Stokes becomes the statement that $\int_S d\omega = \int_{\partial S} \omega$.
Similarly with the Divergence Theorem, $\int_V \nabla \cdot v dV = \oint_{\partial V} v \cdot ds$: associate the vector field $v = (f_x, f_y, f_z)$ with the differential form $\omega = f_x dy \wedge dz + f_y dz \wedge dx + f_z dx \wedge dy$, then $d\omega = \left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} \right) dx \wedge dy \wedge dz$ which we can associate with the scalar field $\nabla \cdot v$. The Divergence Theorem then takes again the form $\int_V d\omega = \int_{\partial V} \omega$.
Even the Fundamental Theorem of Calculus can be brought to bear the same appearance: given a function f, we know that $\int_a^b f'(x)dx = f(b) - f(a)$. In $\mathbb{R}^1$, let $\omega = f$, then $d\omega = \frac{df}{dx} dx$ and if we say $S = [a, b]$ then $\partial S = [b] - [a]$ as a simplicial complex. So the Fundamental Theorem of Calculus becomes $\int_S d\omega = \int_{\partial S} \omega$
To generalise integration to singular simplices or chains, we need to consider the chain

complex $C^{S\infty}_\bullet(M)$ of C^∞ singular simplices as we are working with C^∞ manifolds. It will also be more convenient to work with coefficients in $\mathbb{R}$ rather than in $\mathbb{Z}$ for reasons that will become apparent with de Rham's Theorem.[34] The notation for these groups is $C^{S\infty}_\bullet(M;\mathbb{R})$, and $C^{S\infty}_\bullet(M;\mathbb{Z}) = C^{S\infty}_\bullet(M)$.
Integration then induces a pairing between differential k-forms and C^∞ singular homology as we can pullback integration on M to integration on Δ^k with $\sigma\colon \Delta^k \to M$ a C^∞ map, as follows:

Definition 3.21 For $\omega \in \Omega^k(M)$ and σ a (C^∞) singular k-simplex, define

$$\int_\sigma \omega = \int_{\Delta^k} \sigma^*(\omega)$$

where $\int_{\Delta^k} \sigma^*(\omega)$ is the usual integration of differential forms on $\mathbb{R}^n$.[35]

It is then possible to integrate differential k-forms on C^∞ singular k-chains simply by $\int_{\sum_i a_i\sigma_i} \omega = \sum_i a_i \int_{\sigma_i} \omega$.
First we would like to verify this definition satisfies the expected properties of integration, as done in [Spi99].

Proposition 3.22 Let $\sigma\colon \Delta^k \to \mathbb{R}^k$ be a bijective (C^∞) singular k simplex with $\det(J(\sigma)) \geqslant 0$ on Δ^k (where $J(\sigma)$ is the Jacobian matrix of σ), with ω the differential k-form $\omega = f dx_1 \wedge \cdots \wedge dx_k$ on $\mathbb{R}^k$. Then

$$\int_\sigma \omega = \int_{\sigma(\Delta^k)} f dx_1 \cdots dx_k$$

Proof. Omitted, see [Spi99]. ❖

Corollary 3.23 Let $p\colon \Delta^k \to \Delta^k$ be a (C^∞) diffeomorphism with $\det(J(p)) \geqslant 0$ on Δ^k. For $\sigma\colon \Delta^k \to M$ and $\omega \in \Omega^k(M)$, we have

$$\int_{\sigma\circ p} \omega = \int_\sigma \omega$$

[34]That is, in the definition of a singular chain, replace the coefficients in $\mathbb{Z}$ by coefficients in $\mathbb{R}$. This gives a vector space structure to $C^{S\infty}_k(M;\mathbb{R})$ akin to that of $\Omega^k(M)$

[35]That is, if $\omega = f dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$, $\int_{\Delta^k} \omega = \int_{\Delta^k} f(x_{i_1}, \dots, x_{i_k}) dx_{i_1} dx_{i_2} \dots dx_{i_k}$. We have to insist $i_1 < i_2 < \cdots < i_k$ as $dx_1 \wedge dx_2 = -dx_2 \wedge dx_1$, so if we did not insist that, for $\omega = dx_1 \wedge dx_2$, we would have $\int_\sigma \omega = \int_{\Delta^2} dx_1 dx_2 = \int_{\Delta^2} dx_2 dx_2 = \int_\sigma -\omega$ or $\int \omega = 0$, and the same would happen for all differential k-forms for $k \geqslant 2$. In effect, we want to keep track of orientation before the pullback, because the standard integral on $\mathbb{R}^n$ doesn't depend on it by Fubini's Theorem.

Proof.

$$\begin{aligned}\int_{\sigma\circ p}\omega &= \int_{\Delta^k}(\sigma\circ p)^*(\omega)\\ &= \int_{\Delta^k}p^*(\sigma^*(\omega))\\ &= \int_{\Delta^k}\sigma^*(\omega)\quad \text{By the proposition}\\ &= \int_{\sigma}\omega\end{aligned}$$

This shows that the value of the integral is indeed independent of reparametrisation, as we would hope. ❖

We are now ready for the full generalisation of the above integral theorems:

Theorem 3.24 (Stokes)
Let ω be a differential $(n-1)$-form and c a C^∞ singular n-chain. Then

$$\int_c d\omega = \int_{\partial c}\omega$$

Proof. First we consider the special case when ω is a $(k-1)$-form on $\Delta^k \subset \mathbb{R}^k$.[36] In this case, ω is a sum of forms of the type $f dx_1 \wedge \cdots \hat{dx_i} \cdots dx_k$, so by linearity we may assume, without loss of generality, that $\omega = f dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots dx_k$. Then $d\omega = (-1)^{i+1}\frac{\partial f}{\partial x_i}dx_1 \wedge \cdots \wedge dx_k$. We also have that $\partial\Delta^k$ is given by $\sum_{j=0}^k(-1)^j r_j$ where r_j are the restriction maps $r_j\colon \Delta^{k-1} \to \Delta^k$ given by $r_0(x_1,\ldots,x_{k-1}) = (1-\sum_{j=1}^k x_j, x_1,\ldots,x_{k-1})$ and $r_j = (x_1,\ldots,x_{j-1},0,x_j,\ldots,x_{k-1})$ for $1 \leqslant j \leqslant k$. Hence $\int_{\Delta^k} d\omega = (-1)^{i+1}\int_{\Delta^k}\frac{\partial f}{\partial x_i}dx_1 \wedge \cdots \wedge dx_k$ and $\int_{\partial\Delta^k}\omega = \sum_{j=0}^n(-1)^j\int_{\Delta_{k-1}} r_j^*(\omega)$ which simplifies to just

$$\begin{aligned}&(-1)^i\int_{\Delta_{k-1}} f(x_1,\ldots,x_{i-1},0,x_i,\ldots,x_{k-1})dx_1\cdots dx_{k-1}\\ &+(-1)^{i-1}\int_{\Delta_{k-1}} f(1-\sum_{j=1}^k x_j, x_1,\ldots,x_{k-1})dx_1\cdots dx_{k-1}\end{aligned}$$

[36] For convenience in calculations, note that we are considering $\Delta^k \subset \mathbb{R}^k$, which is given by $(x_1,\ldots,x_k)$ with $0 \leqslant x_j \leqslant 1$ and $\sum_{j=1}^k x_j \leqslant 1$.

We also have that, using Fubini's Theorem and the Fundamental Theorem of Calculus

$$\begin{aligned}\int_{\Delta^k} \frac{\partial f}{\partial x_i} dx_1 \wedge \cdots \wedge dx_k &= \int_{\Delta^{k-1}} \left(\int_0^{1-\sum_{i\neq j} x_j} \frac{\partial f}{\partial x_i} dx_i \right) dx_1 \cdots \hat{dx_i} \cdots dx_k \\ &= \int_{\Delta^{k-1}} \Big(f(x_1, \ldots, x_{j-1}, 1 - \sum_{i \neq j} x_j, x_{j+1}, \ldots, x_k) \\ &\quad - f(x_1, \ldots, x_j, 0, x_{j+1}, \ldots, x_k) \Big) dx_1 \cdots \hat{dx_i} \cdots dx_k\end{aligned}$$

where the integral over Δ^{k-1} is understood to be obtained by omitting the x_j direction in Δ^k.
Now consider the diffeomorphism ψ of $\mathbb{R}^{k-1}$ given by

$$\psi(x_1, \ldots, x_{k-1}) = (x_2, \ldots, x_{i-1}, 1 - \sum_{j=1}^{k-1}, x_i, \cdots, x_{k-1})$$

we see that the Jacobian of this transformation has determinant $(-1)^{i-1}$ so that applying it does not change the value of the integral.
Hence we have that

$$\begin{aligned}\int_{\partial\Delta^k} \omega &= (-1)^i \int_{\Delta_{k-1}} f(x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_{k-1}) dx_1 \cdots dx_{k-1} \\ &+ (-1)^{i-1} \int_{\Delta_{k-1}} f(1 - \sum_{i\neq j} x_j, x_1, \ldots, x_{k-1}) dx_1 \cdots dx_{k-1} \\ &= (-1)^i \int_{\Delta_{k-1}} f(x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_{k-1}) dx_1 \cdots dx_{k-1} \\ &+ (-1)^{i-1} \int_{\Delta_{k-1}} f(x_1, \ldots, x_{i-1}, 1 - \sum_{i\neq j}^{k} x_j, x_i, \ldots, x_{k-1}) dx_1 \cdots dx_{k-1} \\ &= \int_{\Delta_k} d\omega\end{aligned}$$

and this completes the proof of the special case.
Now, for any singular k-simplex c, we have that $\int_{\partial c} \omega = \int_{\partial\Delta^k} c^*(\omega)$. Then

$$\int_{\partial c} \omega = \int_{\partial\Delta^k} c^*(\omega) = \int_{\Delta^k} dc^*(\omega) = \int_{\Delta^k} c^*(d\omega) = \int_c d\omega$$

which shows that the theorem holds for any singular simplex. By linearity, it also holds for any singular chain. ❖

This shows that integration is well defined in terms of cohomology: given $[\omega] \in H^k(M)$ and $[c] \in H_k(M)$, $\int_c \omega + d\psi = \int_c \omega + \int_{\partial c} \psi = \int_c \omega$ as c is a cycle, so that the result of the integral doesn't depend on which differential form we use to represent the equivalence

class. The same remark applies for homology, as for $\int_{c+\partial s}\omega = \int_c \omega + \int_s d\omega = \int_c \omega$ as ω is closed.
This machinery allows us to define integration over a manifold, but first we need the idea of an orientable manifold: we say an n dimensional manifold M is orientable iff there exists a differential n-form ω defined on M that is nowhere 0. This comes from the intuitive idea that on an orientable (connected) manifold, if we define a smooth choice of ordered basis for T_xM, we cannot get from a basis in one orientation to one in another.[37] If the manifold isn't orientable, we can move a positively oriented basis into a negatively oriented basis, and ω has to take opposite signs, so by the Intermediate Value Theorem $\omega = 0$ somewhere "in between". A manifold with a chosen orientation is called oriented.

Proposition 3.25 Let M be an oriented n dimensional manifold, $c_1\colon \Delta^n \to M$ and $c_2\colon \Delta^n \to M$ two singular simplices which are diffeomorphisms onto their image and orientation preserving (with respect to the orientation on M and the standard orientation on $\mathbb{R}^n$). Let ω be a differential form with $\mathrm{supp}(\omega) \subset (c_1(\Delta^n) \cap c_2(\Delta^n))$.
Then $\int_{c_1}\omega = \int_{c_2}\omega$.

Proof. As noted in [Spi99], this follows from Corollary 3.23: it can be slightly modified to tell us that

$$\int_{c_2}\omega = \int_{c_2\circ(c_2^{-1}\circ c_1)}\omega = \int_{c_1}\omega$$

where the modification is just that $c_2^{-1}\circ c_1$ isn't everywhere defined, but because $\mathrm{supp}(\omega) \subset (c_1(\Delta^n) \cap c_2(\Delta^n))$, the result still applies. ❖

This shows us that the integral is independent of which singular simplex we choose (as long as it satisfies the conditions), so we will write this integral $\int_M \omega$.
We can now define the integral of a compactly supported differential form ω on any oriented n-manifold: we just take an open cover $\mathcal{U}$ of $\mathrm{supp}(\omega)$ so that each $U \in \mathcal{U}$ is contained in $c(\Delta^n)$ for some smooth singular simplex c which is an orientation preserving diffeomorphism. We then define $\int_M \omega = \sum_{f\in F}\int_M f\omega$ where F is a (smooth) partition of unity subordinate to the open cover $\mathcal{U}$. We can easily see that Stokes' Theorem still holds, as it suffices to apply it in the individual integrals, with just some minor adjustments to take into account the presence of the partition of unity.
In particular, when M is compact and orientable, we see that integrating a differential form over the whole manifold gives us a real number. In fact, an orientation of M gives us a top degree differential form ω whose integral over the whole manifold is 1 (or some nonzero constant, but we can just divide through), this allows us to pick out an element $[c] \in H_n(M)$ satisfying $\int_c \omega = 1$, called the fundamental class of M, which then generates $H_n(M)$. We will explore further this duality in section 4.9.

[37]Here, say $dx_1, dx_2, \ldots, dx_n$ is a positively oriented basis, and given any permutation $\sigma \in S_n$, the sign of the orientation of $dx_{\sigma(1)}, dx_{\sigma(2)}, \ldots, dx_{\sigma(n)}$ is $\mathrm{sign}(\sigma)$

The de Rham cohomology of a manifold is defined in terms of differential forms, and as such we would expect $H^\bullet_{dR}$ to depend on this differentiable structure on M. We in fact know that de Rham cohomology is homotopy invariant, but much more can be said, as the following theorem shows:

Theorem 3.26 (de Rham)
$H^k_{dR}(M) \cong H^k_{S\infty}(M;\mathbb{R})$ for all k.

Proof. Omitted. We need to consider $I_k \colon \Omega^k \to C^k$ defined by $I(\omega_k)(c_k) = \int_{c_k} \omega_k$ for all k. This map is a cochain map as $I_k(d\omega_{k-1})(c_k) = \int_{c_k} d\omega_{k-1} = \int_{\partial c_k} \omega_{k-1} = I_{k-1}(\omega_{k-1})(\partial c_k)$ by Stokes' Theorem, so $I \circ d = \partial \circ I$ and the diagram in the definition of a (co)chain map is indeed commutative. Therefore I induces a homomorphism in cohomology $I^* \colon H^\bullet_{dR} \to H^\bullet_{S\infty}$. The proof that this homomorphism is in fact an isomorphism can be found in [Lee00], it involves building up spaces from "de Rham spaces", which are spaces in which I is an isomorphism; starting from convex sets by the Poincaré Lemma, all the way up to any manifold, thus showing that I is an isomorphism for all manifolds. ❖

In fact, the map I also preserves the ring structure of both cohomologies: the de Rham cohomology is equipped with the wedge product, and the singular cohomology also comes equipped with a natural product, the cup product $\smile$ given by $(\varphi \smile \psi)(\sigma) = \varphi(\sigma|[v_0, \dots, v_k])\psi(\sigma|[v_k, \dots, v_{k+l}])$.

4 Applications of homology and cohomology

4.1 Degree of a continuous map

For continuous maps $f\colon S^1 \to S^1$, as we know that $\pi_1(S^1)$ (or $H_1(S^1)$) is $\mathbb{Z}$, so we have a notion of how many times f wraps the circle around itself. For example, considering S^1 as $\{z \in \mathbb{C} : |z| = 1\}$, we have the maps $z \mapsto z^n$ which have winding number n; and because $\pi_1(S^1) \cong \mathbb{Z}$ we see that any map $S^1 \to S^1$ is completely classified up to homotopy by its winding number.
We would like to generalise this idea for maps $f\colon S^n \to S^n$. We know that such a map induces a homomorphism $f_*\colon H_n(S^n) \to H_n(S^n)$. As $H_n(S^n) = \mathbb{Z}$, $f_*\colon \mathbb{Z} \to \mathbb{Z}$ is a homomorphism, so must be of the form $f_*(z) = nz$ for some $n \in \mathbb{Z}$ depending only on f. This integer n is called the degree of f and denoted $\deg(f)$.
There is another definition of degree for connected orientable manifolds, given for example in [Hir94]: suppose M and N are compact oriented manifolds of dimensions $m = n$, and suppose N is connected. If we have a map $f\colon M \to N$, and $x \in M$ is a regular point of f with value y, we define $\deg_x(f)$ to be 1 if the map $f_*\colon T_xM \to T_yN$ preserves orientation, and -1 if this map reverses orientation.[38] Now for any regular value y of f, define $\deg(f, y) = \sum_{x \in f^{-1}(y)} \deg_x(f)$. This degree can be interpreted geometrically: $\deg(f, y) = p - q$ where p is the number of preimages of y which are mapped to y in an orientation preserving way, and q is the number of preimages of y which are mapped to y in an orientation reversing manner.
The first definition of degree can also be generalised to manifolds, as done in [Vic94]:
For a manifold M of dimension n, we have that:

$$\begin{aligned} H_k(M, M \setminus \{p\}) &\cong H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \quad \text{by excision} \\ &\cong \widetilde{H}_{k-1}(\mathbb{R}^n \setminus \{0\}) \\ &\cong \widetilde{H}_{k-1}(S^{n-1}) \end{aligned}$$

Where the second step follows by considering the long exact sequence in (reduced) relative homology:

$$\widetilde{H}_k(\mathbb{R}^n) \to \widetilde{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \to \widetilde{H}_{k-1}(\mathbb{R}^n \setminus \{0\}) \to \widetilde{H}_{k-1}(\mathbb{R}^n)$$

because $\widetilde{H}_k(\mathbb{R}^n) = 0$ for all k, so that we get an isomorphism $\widetilde{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong \widetilde{H}_{k-1}(\mathbb{R}^n \setminus \{0\})$.

[38] That is, the map f^* induced by f on top-degree differential forms takes ω to η up to multiplication by an always positive function g on N. This is because, given any two orientations η_1 and η_2, we can define their ratio $\frac{\eta_1}{\eta_2}$ by $\frac{\eta_1}{\eta_2}(p) = \frac{\eta_1(p)}{\eta_2(p)}$ which is a nowhere vanishing function on N, so orientations are unique up to multiplication by nowhere vanishing functions on the manifold. Two orientations are then equivalent if their ratio is an always positive function.

This shows that $H_k(M, M \setminus \{p\})$ is $\mathbb{Z}$ for $k = \dim(M)$ and 0 otherwise. In general, by excision, we can see $H_k(M, M \setminus A)$ as a "local homology" of M around A.
Following [Hat02], an R-orientation of M is a choice of generator of $H_n(M, M \setminus \{p\}; R)$ for all $p \in M$, such that around each point p there is an open ball in which the choice is always the same. This really means that a choice of generator is locally a choice of orientation, but we need to keep that orientation consistent. We can see that this homological definition of orientation is consistent with the version with differential forms, as we can just integrate the differential form on the singular cohomology class and see if we get a positive or negative answer.

Theorem 4.1 Let M be a compact, connected manifold without boundary of dimension n.
If M is R-orientable, then the map : $H_n(M) \to H_n(M, M \setminus \{p\}) \cong R$ is an isomorphism for all $p \in M$.
If M is not R-orientable, then the map : $H_n(M) \to H_n(M, M \setminus \{p\}) \cong R$ is injective, with image $\{r \in R : 2r = 0\}$, for all $p \in M$.

Proof. See [Hat02], [Vic94] or [Pra94]. ❖

In particular, we see that a connected orientable manifold M of dimension n has that $H_n(M) \cong \mathbb{Z}$. In fact, an orientable manifold is R-orientable for all R,[39] so that $H_n(M; R) \cong R$. On the other hand, for a (connected) nonorientable manifold N we see that $H_n(N) \cong 0$ but $H_n(N, \mathbb{Z}_2) \cong \mathbb{Z}_2$. This explains a bit further the terminology "fundamental class": a fundamental class $[M] \in H^n(M)$ is a choice of generator for the top homology whose image is a generator of all the local orientations; the theorem shows that this is possible if and only if M is orientable (and compact, without boundary). The same remark applies for R-orientations.
Hence we can define degree of a smooth map between compact connected orientable[40] manifolds of the same dimension as we could for maps of spheres as the homology groups $H_n(M)$ are $\mathbb{Z}$.[41]

Theorem 4.2 The two definitions of degree of a smooth map between manifolds coincide.

Proof. [Mon06] gives us a lemma that allows us to prove this theorem.

[39]This is because, $H_n(M; R) = H_n(M) \otimes R$, so that if M is $\mathbb{Z}$-orientable, $H_n(M; R) \cong R$.
[40]In fact, it is also possible to do this with nonorientable manifolds by considering homology with coefficients in $\mathbb{Z}_2$, and in this case with the second approach the degree is given by $p - q = p + q \pmod 2$. This is motivated by the above remark that said that $H_n(N, \mathbb{Z}_2) \cong \mathbb{Z}_2$ so that $\mathbb{Z}_2$ becomes the natural coefficient ring to consider.
Also note that the same results hold for manifolds with boundary M, considering $H_n(M, \partial M; R)$ instead (but in this case $H_n(M) \cong 0$).
[41]In fact, even if M isn't compact, we can use the compactly supported differential forms $H^n_c(M)$ to define this degree, as we will see in section 4.9 that $H^n_c(M) \cong \mathbb{Z}$.

Lemma 4.3 (Stack of records)
Suppose $y \in N$ is a regular value of the proper map[42] $f\colon M \to N$. Then there exists a neighborhood $V \subset N$ of y such that $f^{-1}(y) = \cup_i U_i$ with $U_i \cap U_j = \emptyset$ for $i \neq j$, and $f|_{U_i}\colon U_i \to V$ a diffeomorphism for all i.

Proof. See [MT97]. ❖

It will be more convenient to work with the de Rham cohomology than with the singular homology. By de Rham's Theorem, we can consider

$$\begin{array}{ccc} H^n_{dR}(N) & \xrightarrow{f^*} & H^n_{dR}(M) \\ \downarrow I & & \downarrow I \\ \mathbb{R} & \xrightarrow{\deg(f)} & \mathbb{R} \end{array}$$

where we have replaced evaluation against the fundamental class with integration over the whole manifold; we know that this diagram commutes by the definition of $\deg(f)$ (and, for example, using Theorem 4.12).
Hence given a smooth map $f\colon M \to N$ between two compact[43] connected orientable manifolds of dimensions $m = n$ and a differential n-form on N, we have that $\deg(f)\int_N \omega = \int_M f^*\omega$ as the diagram commutes, a generalisation of the special case for f a diffeomorphism.[44]
Then, using the same notation as for the lemma, take $\omega \in \Omega^n(N)$ and write $f_i^*(\omega) = (f|_{U_i})^*(\omega_i)$ where $\sum_i \omega_i = \omega$ is such that the ω_i have support in $f(U_i)$. f is necessarily proper as M is compact, so the lemma tells us:
$\int_M f^*(\omega) = \sum_i \int_{U_i} f_i^*(\omega) = \sum_{x \in f^{-1}(y)} \deg_x(f) \int_V \omega = \deg(f,y)\int_N \omega$.
This shows that the two definitions of degree are indeed equivalent. ❖

In particular, we note that this proves many properties of the original definition $\deg(f,y)$: it is independent of the regular value y that is chosen, and it also only depends on the homotopy type of f.

The notion of degree then allows us to make explicit the boundary maps in cellular homology.

Theorem 4.4 (Cellular boundary formula)
For $n \geqslant 2$, the boundary map $d_n\colon H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$ takes the form $d_n(e_i^n) = \sum_j \deg(f_{ij}) e_j^{n-1}$
where f_{ij} is the composite map $S_i^{n-1} \to X^{n-1} \to S_j^{n-1}$ of the attaching map of e_i^{n-1} with

[42] A proper map is defined by the fact that the preimage of a compact set is compact.
[43] In fact, this can be generalised to the case where M is non-compact, it suffices to consider compactly supported cohomology, and then we also have $H^n_c(M) \cong \mathbb{R}$ even for non-compact (but orientable and connected) M; and we get the same commutative diagram.
[44] In fact we have $\deg(f)\int_N \omega = \int_M f^*\omega$ only for $\omega \in H^n_{dR}(N)$, but we know that because of Stokes' Theorem, we can generalise this to all differential forms (and similarly for the domains of integration).

the quotient map collapsing $X^{n-1} \setminus e_j^{n-1}$ to a point (we have in mind the identification of n-cells of X with generators of the homology $H_n(X^n, X^{n-1})$).

Proof. Following [Hat02], we consider the diagram

$$\begin{array}{ccccc}
H_n(D_i^n, \partial D_i^n) & \xrightarrow[\cong]{\partial} & \widetilde{H}_{n-1}(\partial D_i^n) & \xrightarrow{(f_{ij})_*} & \widetilde{H}_{n-1}(S_j^{n-1}) \\
\downarrow (\Phi_i)_* & & \downarrow (\varphi_i)_* & & \uparrow (q_j)_* \\
H_n(X^n, X^{n-1}) & \xrightarrow{\partial_n} & \widetilde{H}_{n-1}(X^{n-1}) & \xrightarrow{q_*} & \widetilde{H}_{n-1}(X^{n-1}/X^{n-2}) \\
& \searrow^{d_n} & \downarrow a_{n-1} & & \downarrow \cong \\
& & H_{n-1}(X^{n-1}, X^{n-2}) & \xrightarrow{\cong} & H_{n-1}(X^{n-1}/X^{n-2}, X^{n-2}/X^{n-2})
\end{array}$$

where Φ_i is the characteristic map of e_i^n, φ_i is its attaching map, $q\colon X^{n-1} \to X^{n-1}/X^{n-2}$ is the quotient map, $q_j\colon X^{n-1}/X^{n-2} \to S_j^{n-1}$ is the quotient map collapsing $X^{n-1} \setminus e_j^{n-1}$ to a point, and $f_{ij} = q_j \circ q \circ \varphi_i$. Note that a_{n-1} is just the map in the definition of cellular homology.

Notice that the diagram commutes, by definition of the maps involved (for example, commutativity of the bottom left triangle follows from the definition of cellular homology). Now, $(\Phi_i)_*$ takes a chosen generator $[D_i^n] \in H_n(D_i^n, \partial D_i^n)$ to a generator of the $\mathbb{Z}$ factor in $H_n(X^n, X^{n-1})$ corresponding to the cell e_i^n, which we will also write as e_i^n. Because the left half of this diagram commutes, we get that $d_n(e_i^n) = a_{n-1} \circ (\varphi_i)_* \circ \partial[D_i^n]$. Also, $(q_j)_*$ is the projection of $H_{n-1}(X^{n-1}/X^{n-2})$ onto its $\mathbb{Z}$ factor corresponding to the $(n-1)$-cell e_j^{n-1}. Then, because the diagram commutes, we get that $d_n(e_i^n) = q_* \circ (\varphi_i)_* \circ \partial[D_i^n]$, and the previous remark shows that $q_* \circ (\varphi_i)_*$ just gives $\deg(f_{ij})e_j^{n-1}$ on each factor of $\mathbb{Z}$ in $H_{n-1}(X^{n-1}/X^{n-2})$ corresponding to the cell e_j^{n-1}. This shows that $d_n(e_i^n) = \sum_j \deg(f_{ij})e_j^{n-1}$. ❖

4.2 Linking numbers

Given two knots $S^1 \to \mathbb{R}^3$, the idea of degree allows us to compute a linking number, which expresses how the two knots are linked together.

A knot is an embedding $S^1 \to \mathbb{R}^3$, which means that it's a homeomorphism onto its image. Sometimes we require that this extends, to allow for an embedding of $S^1 \times D^2 \to \mathbb{R}^3$, so that we can "thicken" the knot; such knots are called tame. Now, if we are given two disjoint knots $K_1\colon S^1 \to \mathbb{R}^3$ and $K_2\colon S^1 \to \mathbb{R}^3$, we can consider this as a map $p\colon S^1 \times S^1 \to \mathbb{R}^3$ by $p(u,v) = K_1(u) - K_2(v)$. Because the knots are disjoint, this is never 0, so we have a well defined map $q\colon S^1 \times S^1 \to S^2$ given by $q(u,v) = \frac{p(u,v)}{\|p(u,v)\|}$. We call the degree of this map q, from the torus to the sphere, the linking number $\ell(K_1, K_2)$. Note that this depends on choosing orientations, especially on the sphere, so that this is really well defined only up to sign.

In particular, because the degree is defined on the homology level, it is homotopy invariant: it does not change if we can deform one link into another without ever crossing the knots. For example, if we can untangle the two knots, we can easily visualise that the map q is homotopic to a constant map (just put the two knots very far away from each other, for example), so that the linking number is 0, as we would expect. We will give another, easier to compute, description of the linking number in section 4.9.

4.3 Hairy Ball Theorem

The notion of degree allows us to prove many results relatively easily. As we saw above, $\deg(\mathrm{id}) = 1$. On S^n, write $-\mathrm{id}$ for the antipodal map sending p to $-p$ in $\mathbb{R}^{n+1}$. Then $-\mathrm{id}$ just changes the sign of the $n+1$ coordinates (and reverses orientation each time), so $\deg(-\mathrm{id}) = (-1)^{n+1}$. These few facts about the degree allows us to prove the following theorem:

Theorem 4.5 (Hairy Ball Theorem)
There is a nowhere zero (continuous) tangent vector field on S^n if and only if n is odd.

Proof. Suppose X is a nowhere zero (continuous) tangent vector field on S^n. It is nowhere zero so we get an unit tangent vector field $Y = X/\|X\|$ at each point. We also have the normal vector field Z given at each point p by the vector from the origin to p in $\mathbb{R}^{n+1}$. We can then construct a homotopy between the identity and the antipodal map, by setting $f(x,t) = \cos(t)X + \sin(t)Z$ for $0 \leqslant t \leqslant \pi$. Therefore $\deg(\mathrm{id}) = \deg(-\mathrm{id})$. But $\deg(-\mathrm{id}) = (-1)^{n+1}$ so n must be odd.
Conversely, suppose $n = 2k-1$ is odd, then we can directly construct a nowhere zero continuous tangent vector field: $X = (-x_2, x_1, -x_4, x_3, \ldots, -x_{2k}, x_{2k-1})$. $\langle X, Z\rangle = 0$ so this is indeed a tangent vector field. ❖

4.4 Brouwer's Fixed Point Theorem

Homology also allows us to prove the famous Fixed Point Theorem of Brouwer.
First, we need the concept of retractions: a retraction r of X onto Y is a map $r\colon X \to Y$ that satisfies $r(y) = y$ for all $y \in Y$.

Lemma 4.6 There can be no continuous retraction $D^n \to S^{n-1}$

Proof. Suppose r is such a retraction, and i is the inclusion $S^{n-1} \to D^n$. Then $r \circ i = \mathrm{id}$. Hence $(r \circ i)_*\colon \widetilde{H}_{n-1}(S^{n-1}) \to \widetilde{H}_{n-1}(S^{n-1})$ is just the identity on homology. But $r_*\colon \widetilde{H}_{n-1}(D^n) \to \widetilde{H}_{n-1}(S^{n-1})$ and $i_*\colon \widetilde{H}_{n-1}(S^{n-1}) \to \widetilde{H}_{n-1}(D^n)$ are both the 0 map because $\widetilde{H}_{n-1}(D^n) \cong 0$, which is a contradiction as $\widetilde{H}_{n-1}(S^{n-1}) \cong \mathbb{Z}$. ❖

Theorem 4.7 (Brouwer's Fixed Point Theorem)
Every continuous map $f\colon D^n \to D^n$ has a fixed point.

Proof. Suppose $f\colon D^n \to D^n$ does not have any fixed points. This then induces a map $g\colon D^n \to S^{n-1}$ by constructing the line between x and $f(x)$, and setting $g(x)$ to be the point on S^{n-1} where this line (starting at $f(x)$, towards x) meets S^{n-1}. This is a retraction $D^n \to S^{n-1}$ as for $x \in S^{n-1}$, the line will clearly intersect S^{n-1} at x. But this contradicts Lemma 4.6. ❖

4.5 Betti numbers and Euler characteristic

For a finite CW complex X, we would like to generalise the idea of Euler characteristic as a good topological invariant of our space, which is defined for two dimensional spaces as $\chi(X) = c_0 - c_1 + c_2$ where c_i is the number of i cells of X.

Definition 4.8 The Euler characteristic of a cell complex X is the quantity $\chi(X) = \sum_n (-1)^n c_n$.

$\chi(X)$ is defined in terms of the numbers of cells of X, but X might be a manifold with many different CW approximations. Thankfully $\chi(X)$ can be defined purely in terms of the homology of X, which ensures that it does not depend on which CW structure is chosen in this case, by Theorem 2.24.

Theorem 4.9 $\chi(X) = \sum_n (-1)^n \mathrm{rank}(H_n(X))$[45]

Proof. Suppose $0 \to A \to B \to C \to 0$ is a short exact sequence of finitely generated abelian groups; we know that $\mathrm{rank}(B) = \mathrm{rank}(A) + \mathrm{rank}(C)$.
Following [Hat02], consider $0 \to C_n \to \cdots \to C_1 \to C_0 \to 0$ as a chain complex of finitely generated abelian groups, with cycles Z_n, boundaries B_n and homology groups $H_n = Z_n/B_n$. We then have short exact sequences $0 \to Z_n \to C_n \to B_{n-1} \to 0$ and $0 \to B_n \to Z_n \to H_n \to 0$, giving us $\mathrm{rank}(C_n) = \mathrm{rank}(Z_n) + \mathrm{rank}(B_{n-1})$ and $\mathrm{rank}(Z_n) = \mathrm{rank}(B_n) + \mathrm{rank}(H_n)$. Therefore $\mathrm{rank}(C_n) = \mathrm{rank}(B_n) + \mathrm{rank}(B_{n-1}) + \mathrm{rank}(H_n)$. Hence $\sum_n (-1)^n \mathrm{rank}(C_n) = \sum_n \mathrm{rank}(H_n)$ as the B_n cancel out, and we are just left with B_{-1} and B_n which are both 0.
We can now apply this result with $C_n = H_n(X^n, X^{n-1})$: this is in bijection with the n cells of X as we have already noted, so this does indeed prove the theorem. ❖

The quantities $b_i(X) = \mathrm{rank}(H_i(X))$ are called the Betti numbers, and they give a quantitative measure of the holes of X. For example, for the triangle whose homology was computed before, we had $b_1(T) = 1$. For the torus, the homology groups are $H_0(S^1 \times S^1) \cong \mathbb{Z}$, $H_1(S^1 \times S^1) \cong \mathbb{Z}^2$, $H_2(S^1 \times S^1) \cong \mathbb{Z}$, so $b_0 = 1$, $b_1 = 2$, $b_3 = 1$ and $1 - 2 + 1 = 0 = \chi(S^1 \times S^1)$.

[45] We can define the rank (of a finitely generated abelian group A) as the number of $\mathbb{Z}$ summands in A, or equivalently as the dimension of $A \otimes \mathbb{Q}$ as a vector space over $\mathbb{Q}$.

4.6 Universal Coefficient Theorem

Another possible consideration is the importance of which coefficient group we pick for our homology. For example, given some homology group $H_k(X;\mathbb{Z}) \cong \mathbb{Z}$, we can safely say that $H_k(X;\mathbb{Q}) \cong \mathbb{Q}$, or for $H_k(X;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_p$ we get $H_k(X,\mathbb{Q}) \cong \mathbb{Q}$, as taking coefficients in $\mathbb{Q}$ kills torsion (elements of finite order). In general, this is just saying that $H_k(X;A) \cong H_k(X;\mathbb{Z}) \otimes_{\mathbb{Z}} A$, and when A is a field this is indeed true.

The problem arising is that taking a tensor product with A does not necessarily preserve exactness of sequences: we say that $\otimes$ is not an exact functor - in fact, it is a "right exact functor", which means that if $A \to B \to C \to 0$ is exact, then $A \otimes G \to B \otimes G \to C \otimes G \to 0$ is exact, but it is not left exact as $0 \to A \to B \to C$ being exact does not necessarily imply that $0 \to A \otimes G \to B \otimes G \to C \otimes G$ is exact. We can take homology to be a measure of how a sequence fails to be exact, and this means that the relation between homology with different coefficients will not be as simple as taking tensor products, as we get "by-products" to account for loss of exactness.
Given the exact sequence $0 \to B_n(C) \xrightarrow{i_n} Z_n(C) \to H_n(C) \to 0$, $0 \to \ker(i_n \otimes \mathrm{id}) \to A \otimes G \xrightarrow{i_n \otimes \mathrm{id}} B \otimes G \to C \otimes G \to 0$ is exact. In fact $\ker(i_n \otimes \mathrm{id})$ only depends on $H_n(C)$ and G; to see this, consider "free resolutions" of $H_n(C)$, that is, exact sequences of free abelian groups $\cdots \to F_n \to \cdots \to F_1 \to F_0 \to H_n(C) \to 0$, the sequence $0 \to B_n(C) \xrightarrow{i_n} Z_n(C) \to H_n(C) \to 0$ is therefore a free resolution of $H_n(C)$. If we then tensor a free resolution of $H_n(C)$ by G, we no longer get an exact sequence, but it is still a complex, so we can consider the homology $H_n(F \otimes G)$ for the homology of the previous sequence at F_n, but with the term $H_n(C)$ deleted. This in fact does not depend on the resolution we choose, only on $H_n(C)$ and G, so we write it as $\mathrm{Tor}_n(H_n(C), G)$ (we then see why we deleted $H_n(C)$; if we didn't, $\mathrm{Tor}_0(H_n(C), G)$ would be 0). The above free resolution $0 \to B_n(C) \xrightarrow{i_n} Z_n(C) \to H_n(C) \to 0$ then shows that $\mathrm{Tor}_n(H_n(C), G) \cong 0$ for $n > 1$.[46] For this reason, we will often write Tor for Tor_1.
When G is any abelian group, the natural map $H_k(X;\mathbb{Z}) \otimes G \to H_k(X;G)$ is not necessarily surjective anymore, but we can precisely say how far it is from being surjective:

Theorem 4.10 (Universal Coefficient Theorem for homology)
Let G be any abelian group. Then homology with coefficients in G is related to homology with coefficients in $\mathbb{Z}$ by the following short exact sequence:

$$0 \to H_n(X) \otimes G \to H_n(X;G) \to \mathrm{Tor}(H_{n-1}(X), G) \to 0$$

[46]The reason for the notation Tor_n is that these higher Tor functors need not be 0 if we are working in more generality than with abelian groups; for example, if we are working with modules over any ring. In this case, we need to take a projective resolution, not a free resolution (although all free resolutions are automatically projective).

Proof. It suffices to consider the exact sequence of chain complexes

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_{n+1} & \xrightarrow{i_{n+1}} & C_{n+1} & \xrightarrow{\partial_{n+1}} & B_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_{n+1}} & & \downarrow{\scriptstyle \partial_{n+1}} & & \downarrow{\scriptstyle \partial_n} & & \\
0 & \longrightarrow & Z_n & \xrightarrow{i_n} & C_n & \xrightarrow{\partial_n} & B_{n-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle \partial_{n-1}} & & \\
0 & \longrightarrow & Z_{n-1} & \xrightarrow{i_{n-1}} & C_{n-1} & \xrightarrow{\partial_{n-1}} & B_{n-2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

where obviously the boundary maps are 0 for Z_n and B_n. Tensoring it with G, we get

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_{n+1}\otimes G & \xrightarrow{i_{n+1}\otimes \mathrm{id}} & C_{n+1}\otimes G & \xrightarrow{\partial_{n+1}\otimes \mathrm{id}} & B_n\otimes G & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_{n+1}\otimes \mathrm{id}} & & \downarrow{\scriptstyle \partial_{n+1}} & & \downarrow{\scriptstyle \partial_n\otimes \mathrm{id}} & & \\
0 & \longrightarrow & Z_n\otimes G & \xrightarrow{i_n\otimes \mathrm{id}} & C_n\otimes G & \xrightarrow{\partial_n\otimes \mathrm{id}} & B_{n-1}\otimes G & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_n\otimes \mathrm{id}} & & \downarrow{\scriptstyle \partial_n\otimes \mathrm{id}} & & \downarrow{\scriptstyle \partial_{n-1}\otimes \mathrm{id}} & & \\
0 & \longrightarrow & Z_{n-1}\otimes G & \xrightarrow{i_{n-1}\otimes \mathrm{id}} & C_{n-1}\otimes G & \xrightarrow{\partial_{n-1}\otimes \mathrm{id}} & B_{n-2}\otimes G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

in fact this is still a short exact sequence of chain complexes because we had sequences $0 \to Z_n \to C_n \to B_{n-1} \to 0$ that were split, which means that $C_n \cong Z_n \oplus B_{n-1}$; this is because all groups involved are free abelian. Exactness of the diagram then follows because the tensor product distributes over direct sums. By the Zigzag lemma (Lemma 2.13), this gives us a long exact sequence in homology

$$\cdots \to B_n \otimes G \to Z_n \otimes G \to H_n(C;G) \to B_{n-1} \otimes G \to \cdots$$

where the homologies of B_n and Z_n are just themselves as all the corresponding boundary maps are 0, as noted above. We can change this long exact sequence into short exact sequences, in a way which reverses the weaving process we did earlier for cellular homology, and we get sequences

$$0 \to \text{coker}(i_n \otimes \text{id}) \to H_n(C;G) \to \ker(i_{n-1} \otimes \text{id}) \to 0$$

where the cokernel $\text{coker}(i_n \otimes \text{id})$ is just $(Z_n \otimes G)/\text{im}(i_n \otimes \text{id})$. Because of the free resolution $0 \to B_n \xrightarrow{i_n} Z_n \to H_n \to 0$, we get exactness of $B_n \otimes G \to Z_n \otimes G \to H_n \otimes G \to 0$, so that $\text{coker}(i_n \otimes \text{id}) = H_n(C) \otimes G$ by the First Isomorphism Theorem. The above discussion about the tensor product and Tor then shows that $\ker(i_{n-1} \otimes \text{id}) = \text{Tor}(H_{n-1}(X), G)$, which completes the proof. ❖

Proposition 4.11 $\text{Tor}(A, B) = \text{Tor}(B, A)$
$\text{Tor}(\oplus_i A_i, B) = \oplus_i \text{Tor}(A_i, B)$
$\text{Tor}(T(A), B) = \text{Tor}(A, B)$ where $T(A)$ is the torsion subgroup of A (subgroup consisting of all elements of finite order).
$\text{Tor}(\mathbb{Z}_n, B) = \ker(B \xrightarrow{\times n} B)$
Given an exact sequence $0 \to A \to B \to C \to 0$, we get an exact sequence

$$0 \to \text{Tor}(G, A) \to \text{Tor}(G, B) \to \text{Tor}(G, C) \to A \otimes G \to B \otimes G \to C \otimes G \to 0$$

Proof. Omitted, see [Hat02]. ❖

The previous proposition then explains where Tor takes its name from: torsion. Indeed, $\text{Tor}(A, B)$ is 0 whenever A or B is torsion-free. In particular, this implies that for $\mathbb{Q}$, we do have that $H_k(X; \mathbb{Q}) \cong H_k(X) \otimes \mathbb{Q}$.
There also exists a Universal Coefficient Theorem for cohomology, that stems from a slightly different motivation. We know we can define the cohomology as a formal dual of homology by dualising the chain complex $C^\bullet = \text{Hom}(C_\bullet, \mathbb{Z})$, and we might wonder if this dualisation passes down to (co)homology; is it true that $H^k(X) \cong \text{Hom}(H_k(X), \mathbb{Z})$? This is not always the case, though once again it holds when we consider coefficients in $\mathbb{Q}$. In this case, we are no longer working with the tensor product $\otimes$ but with Hom, which also does not preserve exactness.
Consider the free resolution $0 \to B_n \to C_n \to H_n \to 0$, this gives us the exact sequence $0 \to \text{Hom}(H_n, \mathbb{Z}) \to \text{Hom}(C_n, \mathbb{Z}) \to \text{Hom}(B_n, \mathbb{Z})$. In this case we have a cochain complex, so we chop off the last term and take the cohomology: this gives us the ext functors Ext_n. As before, for abelian groups, we have that $\text{Ext}_n = 0$ for $n > 1$, so we write Ext for Ext_1. This then allows us to state the Universal Coefficient Theorem for cohomology:

Theorem 4.12 (Universal Coefficient Theorem for cohomology)
Let A be any abelian group. Then cohomology with coefficients in A is related to homology with coefficients in A by the following short exact sequence:

$$0 \to \text{Ext}(H_{n-1}(X), A) \to H^n(X; A) \to \text{Hom}(H_n(X), A) \to 0$$

Proof. This is entirely analogous to the proof of the Universal Coefficient Theorem for homology. ❖

This shares many properties with the Universal Coefficient Theorem for homology, but is also dual in some sense: the natural map $H^n(X;A) \to \mathrm{Hom}(H_n(X;A),A)$ is now surjective, but no longer necessarily injective, as its kernel is precisely $\mathrm{Ext}(H_{n-1}(X;A),A)$.

Proposition 4.13 $\mathrm{Ext}(A_1 \oplus A_2, B) = \mathrm{Ext}(A_1,B) \oplus \mathrm{Ext}(A_2,B)$
$\mathrm{Ext}(A,B) = 0 \quad$ if A is free.[47]
$\mathrm{Ext}(\mathbb{Z}_n,B) = G/nG$

Proof. The first of these properties follows because a free resolution of $A_1 \oplus A_2$ can be obtained by taking the direct sum of the individual free resolutions of A_1 and A_2.
If A is free, it has the free resolution $0 \to A \to A \to 0$, giving the second property.
For the third, consider the exact sequence $0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \mathbb{Z}_n \to 0$. Applying Hom, in a similar way than for Tor, we get the following exact sequence:

$$0 \leftarrow \mathrm{Ext}(\mathbb{Z},A) \leftarrow \mathrm{Ext}(\mathbb{Z},A) \leftarrow \mathrm{Ext}(\mathbb{Z}_n,A) \leftarrow \mathrm{Hom}(\mathbb{Z},A) \xleftarrow{\times n} \mathrm{Hom}(\mathbb{Z},A) \leftarrow \mathrm{Hom}(\mathbb{Z}_n,A) \leftarrow 0$$

which reduces to just

$$0 \leftarrow \mathrm{Ext}(\mathbb{Z},A) \leftarrow G \xleftarrow{\times n} G \leftarrow \cdots$$

so by exactness $\mathrm{Ext}(\mathbb{Z},A) \cong G/nG$. ❖

4.7 Kunneth Formula

Given two CW complexes X and Y, we can form their product $X \times Y$ (which does not have the product topology, it has the CW topology associated to the product CW structure); this gives us the "cross product" map $\times\colon C_k(X) \times C_l(Y) \to C_{k+l}(X \times Y)$. This map is compatible with the boundary operator d as $d(e_i \times e_j) = (de_i) \times e_j + (-1)^i e_i \times de_j$ where e_i and e_j are i and j cells, respectively; this is proven in [Hat02]. Hence the product of two cycles is a cycle, and the product of a cycle with a boundary is sent to 0 (hence is a boundary), so that we have a map $\times\colon H_k(X) \times H_l(Y) \to H_{k+l}(X \times Y)$, called the cross-product map.
As the cross product is bilinear, we can in fact consider it as a map $\times\colon H_k(X) \otimes H_l(Y) \to H_{k+l}(X \times Y)$, and by considering this over all $k + l = n$, we get a map $\times\colon \bigoplus_{k=0}^{n} (H_k(X) \otimes H_{n-k}(Y)) \to H_n(X \times Y)$, and we might wonder if this map is an isomorphism, as we are looking at all possible origins of a cycle in $X \times Y$ from products of cycles in X and cycles in Y.
For example, consider $Y = S^m$, then $d(x \times y) = dx \times y$ as $dy = 0$ (this follows from the cell structure of S^m with a 0-cell and a m-cell). Hence the chain complex of $X \times S^m$ is

[47]In fact, this can be considerably strengthened. $\mathrm{Ext}(A,B) = 0$ if A is projective or if B is injective, as $\mathbb{Z}$-modules.

just the direct sum of two copies of the chain complex of X, but with one copy having its indices shifted up by m, as we have one such complex for each nonzero cell of S^m. Therefore $H_n(X \times S^m) \cong H_k(X) \oplus H_{n-k}(S^m)$ which shows that the cross product is an isomorphism.

Another example, from [Hat02], is the following: consider $X = S^1 \sqcup_f D^2$ and $Y = S^1 \sqcup_g D^2$ where $\deg(f) = p$ and $\deg(g) = q$. By Theorem 4.4 we have that $H_1(X) = \mathbb{Z}_p$, $H_1(Y) = \mathbb{Z}_q$, and H_2 is 0 for both spaces. X and Y both have 3 cells, so $X \times Y$ has 9. The cell structure of $X \times Y$ fits in the following diagram:

$$\begin{array}{ccccc}
e_0 \times e_2 & & e_1 \times e_2 & \xleftarrow[p]{} & e_2 \times e_2 \\
\downarrow q & & \downarrow -q & & \downarrow q \\
e_0 \times e_1 & & e_1 \times e_1 & \xleftarrow[p]{} & e_2 \times e_1 \\
 & & & & \\
e_0 \times e_0 & & e_1 \times e_0 & \xleftarrow[p]{} & e_2 \times e_0
\end{array}$$

For instance, the two arrows leaving from the top right corner mean that $d(e_2 \times e_2) = pe_1 \times e_2 + qe_2 \times e_1$.

Therefore we have that $H_1(X \times Y) \cong \mathbb{Z}_p \oplus \mathbb{Z}_q$. For $H_2(X \times Y)$, we have that $\ker(d)$ is generated by $e_1 \times e_1$, and the image is of the form $(ap - bq)(e_1 \times e_1)$, which is cyclic of order $\gcd(p, q) = m$, so that $H_2(X \times Y) \cong \mathbb{Z}_m$. For $H_3(X \times Y)$, cycles must be of the form $\frac{q}{m}(e_2 \times e_1) + \frac{p}{m}(e_1 \times e_2)$, and the smallest multiple of such cycle that is a boundary is $q(e_2 \times e_1) + p(e_1 \times e_2)$ so that $H_2(X \times Y) \cong \mathbb{Z}_m$ again. In this case, we see that $H_2(X \times Y) \cong H_1(X) \otimes H_1(Y)$ but $H_3(X \times Y)$ does not come from the cross-product as X and Y have no nontrivial homology in dimensions 2 and above. This fact is remedied if we take coefficients in a field, then the cross product is then always an isomorphism:

Theorem 4.14 (Kunneth Formula for fields)
Let F be a field. Then the cross product map

$$\times \colon \bigoplus_{k=0}^{n} (H_k(X; F) \otimes H_{n-k}(Y; F)) \to H_n(X \times Y; F)$$

is an isomorphism.

Proof. The proof hinges on the idea of a tensor product of chain complexes. Given two chain complexes (C, ∂_C) and (D, ∂_D), we define the tensor product complex $C \otimes D$ by $(C \otimes D)_n = \bigoplus_{p+q=n} C_p \otimes D_q$, with boundary map $\partial_\otimes(c \otimes d) = \partial_C c \otimes d + (-1)^p c \otimes \partial_D d$

where $c \in C_p$. Then we have that

$$\begin{aligned}\partial_\otimes^2(c \otimes d) &= \partial_\otimes\left(\partial_C c \otimes d + (-1)^p c \otimes \partial_D d\right) \\ &= \partial_C^2 c \otimes d + (-1)^{p-1}\partial_C c \otimes \partial_D d + (-1)^p \partial_C c \otimes \partial_D d + (-1)^{2p} c \otimes \partial_D^2 d \\ &= 0\end{aligned}$$

which shows that the tensor product complex is indeed a chain complex. We can easily see the motivation for this by the above remarks with the cross product map that had a similar boundary operator, and indeed this shows that for CW complexes X and Y, $C(X \times Y) \cong C(X) \otimes C(Y)$, as chain complexes.

Coming back to our case, writing $C_n(X)$ for $C_n(X; F)$ and so on, we have a short exact sequence $0 \to Z_n(X) \to C_n(X) \to B_{n-1}(X) \to 0$ which gives us a short exact sequence of chain complexes as in the proof of the Universal Coefficient Theorem for homology. Take the tensor product of each column with the chain complex $C(Y)$, this preserves exactness of the rows as all spaces considered are vector spaces, so that Tor is 0. This gives us the short exact sequence of chain complexes $0 \to Z_\bullet(X) \otimes C_\bullet(Y) \to C_\bullet(X) \otimes C_\bullet(Y) \to B_\bullet(X) \otimes C_\bullet(Y) \to 0$.

Since the boundary maps in $Z_n(X)$ are 0, the boundary maps in $Z_\bullet(X) \otimes C_\bullet(Y)$ are given by $z \otimes y \mapsto (-1)^d z \otimes \partial y$ where $z \in Z_d(X)$. Hence, passing to homology we get that $H_\bullet(Z_\bullet(X) \otimes C_\bullet(Y)) \cong Z_\bullet(X) \otimes H_\bullet(C_\bullet(Y))$, and similarly $H_\bullet(B_\bullet(X) \otimes C_\bullet(Y)) \cong B_\bullet(X) \otimes H_\bullet(C_\bullet(Y))$.

Now, applying the Zigzag Lemma to the short exact sequence of chain complexes we had gives a long exact sequence

$$\cdots \to Z_\bullet(X) \otimes H_\bullet(C_\bullet(Y)) \to H_\bullet(C_\bullet(X) \otimes C_\bullet(Y)) \to B_\bullet(X) \otimes H_\bullet(C_\bullet(Y)) \to \cdots$$

and tensoring the free resolution $0 \to B_\bullet(X) \to Z_\bullet(X) \to H_\bullet(C_\bullet(X)) \to 0$ with $H_\bullet(Y)$ gives the exact sequences $0 \to B_\bullet(X) \otimes H_\bullet(Y) \to Z_\bullet(X) \otimes H_\bullet(Y) \to H_\bullet(X) \otimes H_\bullet(Y) \to 0$.
Hence we have the following diagram

$$\begin{array}{ccccccc} 0 \longrightarrow \bigoplus_n \bigoplus_{p+q=n} B_p \otimes H_q(Y) & \xrightarrow{\quad i \quad} & \bigoplus_n \bigoplus_{p+q=n} Z_p \otimes H_q(Y) & \xrightarrow{a} & \bigoplus_n \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \longrightarrow 0 \\ \nwarrow d & & \swarrow j & & \\ & \bigoplus_n H_n(\bigoplus_{p+q=n} C_p(X) \otimes C_q(Y)) & & & \end{array}$$

where the row and the triangle are exact. This gives us a sequence

$$0 \to \bigoplus_n \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \xrightarrow{j \circ a^{-1}} \bigoplus_n H_n\left(\bigoplus_{p+q=n} C_p(X) \otimes C_q(Y)\right) \to 0$$

where a^{-1} is the right inverse to a guaranteed by exactness of the row. This is well defined as any choice of right inverse a^{-1} gives the same composite $j \circ a^{-1}$ because $\mathrm{im}(i) \subset \ker(j)$, in fact $\text{im}(i) = \ker(j)$ implies that $j \circ a^{-1}$ is injective; this also shows that to prove $j \circ a^{-1}$ is surjective we only need to show that j is surjective; this is easy as $\text{im}(d) = \ker(i) = 0$ so that $\text{im}(j) = \ker(d)$ and j is surjective. As the maps j and a preserve the grading (they preserve the degree), we get the isomorphism $\bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \xrightarrow{j \circ a^{-1}} H_n(\bigoplus_{p+q=n} C_p(X) \otimes C_q(Y))$. Because $\bigoplus_{p+q=n} C_p(X) \otimes C_q(Y) \cong C_n(X \times Y)$, we get the required isomorphism; we also see that $j \circ a^{-1}$ is indeed the cross product map. ❖

However we are often working with coefficients in $\mathbb{Z}$, which is not a field. It is possible to remedy this, but as shown in the example above, the cross product is no longer an isomorphism; but we can use a Tor functor to measure how far it is from being an isomorphism, in a way that is quite similar to the Universal Coefficient Theorem for homology.

Theorem 4.15 (Kunneth Formula for principal ideal domains)
Let R be a principal ideal domain. We then have the following short exact sequence:

$$0 \to \bigoplus_{k=0}^{n} (H_k(X;R) \otimes H_{n-k}(Y;R)) \xrightarrow{\times} H_n(X \times Y;R) \to \bigoplus_{k=0}^{n} \text{Tor}_R\,(H_k(X;R), H_{n-k-1}(Y;R)) \to 0$$

Proof. Omitted, see [Hat02]. We can use essentially the same technique as above, except that $0 \to B_\bullet(X) \otimes H_\bullet(Y) \to Z_\bullet(X) \otimes H_\bullet(Y) \to H_\bullet(X) \otimes H_\bullet(Y) \to 0$ is no longer exact, and we get a Tor functor coming in. We can then apply the same idea as in the preceding proof with the diagram with an exact row and an exact triangle, and this gives the theorem. ❖

In particular, this shows that the cross product is injective, and its failure to be surjective is measured by $\bigoplus_{k=0}^{n} \text{Tor}_R\,(H_k(X;R), H_{n-k-1}(Y;R))$. When R is in fact a field, this Tor functor will always be 0, thus recovering the above statement about the Kunneth formula for fields. The importance of R being a principal ideal domain is that it means that submodules of free R-modules are also free, and higher Tor functors are 0, as for abelian groups. The general statement of the Kunneth formula is somewhat more complicated, as it involves higher Tor functors and spectral sequences; we will not give it here, but a complete account can be found in [Wei94].

4.8 Classifying spaces and Eilenberg-MacLane spaces

Moore spaces

Given an group G, we would like to construct a topological space which has homology groups related to G. First of all, one might consider a space X such that $H_n(X) \cong G$ and $\widetilde{H}_k(X) \cong 0$ for $k \neq n$. To do this, we had better require that G be abelian.

Proposition 4.16 For any abelian group G and any integer $n > 0$, there exists a space $M(G,n)$ (called an n-th Moore space for G) such that $H_n(M(G,n)) \cong G$ and $\widetilde{H}_k(M(G,n)) \cong 0$ for all $k \neq n$.

Proof. Omitted. Note however that the corresponding result for cohomology is not true. For example, consider a "cohomology $M(G,1)$"; by the Universal Coefficient Theorem for cohomology we have that $H^1(M(G,1)) \cong \operatorname{Hom}(H_1(M(G,1)), \mathbb{Z})$, this means that $H^1(M(G,1))$ is torsion-free, so in particular there is no "cohomology $M(G,1)$" for finite G. Similar arguments using Hom and Ext yield many interesting restrictions on what the cohomology can be. ❖

In particular, we see that this allows us to construct a topological space with any prescribed sequence of abelian groups as its homology: simply take a wedge sum of the corresponding Moore spaces.
It turns out that the Moore spaces are not a particularly canonical construction. For example, if we take any $M(G,n)$ with $n \geqslant 2$ (which just means a space X with $H_n(X) \cong 0$ and $\widetilde{H}_k(X) \cong 0$ for $k \neq n$), we see that we can take the wedge sum of this with a space that has perfect fundamental group[48] and trivial other homology groups (see section 4.8 to see that such a space exists). In this case, both spaces have isomorphic homology groups but are not homotopy equivalent as they have different fundamental groups.

Eilenberg-MacLane spaces

One way to remedy the fact that Moore spaces are not unique up to homotopy equivalence is to consider a homotopical analog, Eilenberg-MacLane spaces. Given a group G, an Eilenberg-MacLane space $K(G,n)$ is a topological space satisfying $\pi_n(K(G,n)) \cong G$ and $\pi_k(K(G,n)) \cong 0$ for all $k \neq n$. Of course, if $n > 1$, we require that G be abelian.

Proposition 4.17 For any integer $n > 0$ and any group G (abelian if $n > 1$), there exists a space $K(G,n)$ satisfying $\pi_n(K(G,n)) \cong G$ and $\pi_k(K(G,n)) \cong 0$ for all $k \neq n$. Furthermore, this space is unique up to weak homotopy equivalence.

Proof. This will follow from the next theorem. ❖

From this proof, we see that we have realised a $K(G,n)$ as a CW complex X. By Theorem 2.23, this shows that any other CW complex Y that is a $K(G,n)$ is in fact homotopy equivalent to X. For this reason we sometimes speak of any such space as $K(G,n)$.
Now consider the case of $G = \mathbb{Z}$. One convenient way to construct $K(\mathbb{Z},1)$ is to realise it as $\mathbb{R}/\mathbb{Z} = S^1$: we see that this provides the required homology as $\mathbb{R}$ is weakly contractible and $\mathbb{Z}$ acts freely on $\mathbb{R}$. We can iterate this construction by considering a weakly contractible

[48] A group is called perfect if it is equal to its own commutator subgroup, so that its abelianisation is the trivial group.

space on which S^1 acts freely, and then consider the quotient under this action. In this case, we get the weakly contractible space S^∞ considered as sequences of complex numbers,[49] and its quotient by S^1 is then $\mathbb{CP}^\infty$. [50] In this case, we see that S^1 is a $K(\mathbb{Z},1)$, and also that $\mathbb{CP}^\infty$ is a $K(\mathbb{Z},2)$ as we know that $\pi_1(\mathbb{CP}^\infty) \cong 0$, $\pi_2(\mathbb{CP}^n) \cong \mathbb{Z}$, $\pi_k(\mathbb{CP}^n) \cong 0$ for $2 < k < 2n-1$ and $\pi_{2n-1}(\mathbb{CP}^n) \cong \mathbb{Z}$.[51] We can then "tend n to ∞"[52] and we get the correct homotopy groups for $\mathbb{CP}^\infty$ to be a $K(\mathbb{Z},2)$.
This procedure is called the classifying space construction: given any topological group G,[53] we consider a weakly contractible space EG on which G acts freely, and take the quotient $BG = EG/G$, called classifying space of G.

Theorem 4.18 Let G be a topological group. Then there exists a contractible space EG on which G acts freely (by continuous action) and a space BG (the classifying space), unique up to homotopy equivalence, such that $BG \cong EG/G$.[54]

Proof. (Sketch) Following [AM94], we construct BG and EG as follows.
Consider Δ^n as $\{(x_1,\dots,x_n) : 0 \leqslant x_1 \leqslant \cdots \leqslant x_n \leqslant 1\}$.
Construct $E = \coprod_{n=0}^\infty G \times \Delta^n \times G^n$, and quotient out by the relations:

$$\begin{aligned}
(g,x_1,\dots,x_n,g_1,\dots,g_n) &\sim (g,x_1,\dots,\hat{x}_i,\dots,x_n,g_1,\dots,\hat{g}_i,\dots,g_n) && \text{if } g_i = e \text{ or } x_i = x_{i+1}\\
(g,x_1,\dots,x_n,g_1,\dots,g_n) &\sim (gg_1,x_2,\dots,x_n,g_2,\dots,g_n) && \text{if } x_1 = 0\\
(g,x_1,\dots,x_n,g_1,\dots,g_n) &\sim (g,x_1,\dots,x_{n-1},g_1,\dots,g_{n-1}) && \text{if } x_n = 1
\end{aligned}$$

We claim that $E/\sim \cong EG$. Indeed, a free action of G on $E/\sim$ is given by

$$h(g,x_1,\dots,x_n,g_1,\dots,g_n) = (hg,x_1,\dots,x_n,g_1,\dots,g_n)$$

[49] Precisely, S^∞ is given by the colimit of the chain of inclusions $S^1 \to S^2 \to \cdots$, which means that S^∞ is the union of all the finite-dimensional spheres. S^∞ is indeed contractible: as [Hat02] notices, we have a homotopy $f\colon \mathbb{C}^\infty \times I \to \mathbb{C}^\infty$ from id to $g\colon (x_1,x_2,\dots) \mapsto (0,x_1,x_2,\dots)$ given by $f_t(x_1,x_2,\dots) = (1-t)\mathrm{id}+tg$, so we then get a homotopy from id to g, now acting on S^∞. Then we have a homotopy $h/|h|$ between g and the constant map given by $h_t = (1-t)g + t(1,0,0,\dots)$, so that the identity is homotopic to the constant map, so S^∞ is contractible.

[50] Again, $\mathbb{CP}^\infty$ is defined more precisely as the infinite union of the $\mathbb{CP}^n$, which is the categorical colimit of the sequence of inclusions $\mathbb{CP}^n \to \mathbb{CP}^{n+1}$.

[51] To see this, consider the fiber bundle $S^1 \to S^{2n-1} \to \mathbb{CP}^n$. It is possible to show that fiber bundles induce a long exact sequence of homotopy groups, in this case we get the long exact sequence $\pi_i(S^1) \to \pi_i(S^{2n-1}) \to \pi_i(\mathbb{CP}^n) \to \pi_{i-1}(S^1) \to \cdots$. For $i > 2, i \neq 2n-1$, exactness implies $\pi_1(\mathbb{CP}^\infty) \cong 0$. For $i = 2$ we have an isomorphism $\pi_2(\mathbb{CP}^\infty) \cong \pi_1(S^1) \cong \mathbb{Z}$, and similarly for $i = 2n-1$.

[52] This is possible by consideration of the construction of $\mathbb{CP}^\infty$ by direct limit of the system of inclusions, as moving up from $\mathbb{CP}^n$ to $\mathbb{CP}^{n+1}$ cannot change the homotopy groups π_k for $k < 2n-2$.

[53] A topological group is simply a group whose elements are points of a topological space, such that the group multiplication and inversion are continuous functions on this topological space. In particular, we can consider any group G as a discrete group by giving it the discrete topology.

[54] The name classifying space of BG comes from consideration of G-bundles: every G-bundle is the pullback bundle of the "universal bundle" $EG \to BG$, so that BG captures all the information there is about G-bundles, it classifies them. In fact, the classifying space construction is functorial, so that for any group homomorphism $G \xrightarrow{f} H$ we get a bundle map $(\pi, G, EG, BG) \xrightarrow{Bf} (\varpi, H, EH, BH)$ so that $B(f \circ g) = Bf \circ Bg$ and $B(\mathrm{id}) = \mathrm{id}$.

the quotients insure that this is indeed free (and it is still well defined).
$E/\sim$ is also contractible, a contraction is given by

$$F_t(g, x_1, \ldots, x_n, g_1, \ldots, g_n) = (e, t, \overline{t + x_1}, \ldots, \overline{t + x_n}, g, g_1, \ldots, g_n)$$

where $\overline{x} = \min(x, 1)$.
Similarly, we construct BG as $\coprod_{n=0}^{\infty} \Delta^n \times G^n$ quotiented out by $\sim$, which are the same relations as before but forgetting about the first term.
It remains to be shown that this is indeed the required bundle and that it satisfies uniqueness up to weak homotopy equivalence; refer to [AM94]. ❖

For a discrete abelian group G, define $K(G, 0) = G$. The above discussion with $B\mathbb{Z} = S^1 = K(G, 1)$ and $B^2\mathbb{Z} = \mathbb{CP}^\infty = K(G, 2)$ hints at the following:

Corollary 4.19 Let G be a discrete group, then $BG \cong K(G, 1)$. If G is abelian, we also have that $BK(G, n) \cong K(G, n + 1)$ for all $n \geqslant 1$.

Proof. Consider the fiber bundle $\xi = (\pi, G, EG, BG)$, as noted earlier, this induces a long exact sequence in homotopy groups

$$\cdots \to \pi_n(G) \to \pi_n(EG) \to \pi_n(BG) \to \pi_{n-1}(G) \to \pi_{n-1}(EG) \to \cdots\cdots$$

as EG is contractible, it has all homotopy groups trivial and this gives us isomorphisms $\pi_n(BG) \cong \pi_{n-1}(G)$, hence $BK(G, n)$ has all homotopy groups shifted up a dimension, and is thus a $K(G, n + 1)$.
It remains to show that for a discrete group, BG is a $K(G, 1)$. In this case, we see that G acts freely and properly discontinuously on EG, and that BG and EG have the same homotopy groups except possibly the fundamental group.[55] As EG is simply connected, we get that the quotient space $EG/G \cong BG$ has fundamental group G (and other homotopy groups trivial), so that BG is a $K(G, 1)$. ❖

Some examples of classifying spaces and Eilenberg-Moore spaces:
$G = \mathbb{Z}_2$. Then $BG = \mathbb{RP}^\infty$. To see this, $\mathbb{RP}^\infty$ is the quotient of $EG = S^\infty$ by the usual action of $\mathbb{Z}_2$, which is the identification of antipodal points. We saw above that S^∞ is contractible.[56]

[55]This follows from the consideration of covering spaces: a covering map (like $\pi\colon EG \to BG$ with G discrete) induces isomorphisms on all homotopy groups except π_1, and all the homotopy groups of EG are 0 as it is contractible.

[56]Computing the cohomology rings of $\mathbb{CP}^\infty$ and $\mathbb{RP}^\infty$ gives great insight into line bundles (rank 1 vector bundles): the fact that they are $B^2\mathbb{Z} = BS^1$ and $B\mathbb{Z}_2$ shows that their cohomology can be pulled back to all other S^1 or $\mathbb{Z}_2$ bundles, of which (complex and real, respectively) line bundles are a particular case (up to homotopy, at least, as $\mathbb{C}^\times \simeq S^1$ and $\mathbb{R}^\times \simeq \mathbb{Z}_2$; to show this is indeed what we need to consider requires the theory of principal bundles and structure groups). This actually gives rise, respectively, to the so called first Chern class $c_1 \in H^2(X, \mathbb{Z})$ and first Stiefel-Whitney class $w_1 \in H^1(X, \mathbb{Z}_2)$ of the line bundles. The same can be done for classifying spaces corresponding to more general vector bundles, and

$G = \mathbb{Z}_n$. In the same way as we did above, we can consider $\mathbb{Z}_n$ acting on $S^\infty \subset \mathbb{C}^\infty$ by action of the n-th roots of unity, the quotient we then get is the infinite dimensional Lens space L_n^∞.[57]

Cohomology spectra

Given a sequence of spaces (K_n) and another space X, consider the homotopy classes of basepoint preserving maps from X to K_n, denoted $\langle X, K_n \rangle$. We would like to define a cohomology theory by $h_n(X) = \langle X, K_n \rangle$. First of all, this requires putting an abelian group structure on $\langle X, K_n \rangle$. Recalling the group structure of homotopy groups, with $\langle S^n, K_n \rangle$, we collapsed S^n along the equatorial S^{n-1} to acquire two copies of S^n, allowing a group structure. In more generality, this works when considering the suspension SX of X, defined as $SX = X \times I / X \times \{0\} \cup X \times \{1\}$, and we can again collapse SX along an equatorial X to get $SX \vee SX$. Care has to be taken, however, as we are considering basepoint preserving maps, and it may not be entirely clear what basepoints to take in that case, so instead we consider the reduced suspension ΣX given by $\Sigma X = X \times I / X \times \{0\} \cup \{p\} \times I \cup X \times \{1\}$ where p is the designated basepoint of X. This then makes $\langle \Sigma X, K_n \rangle$ into a group. However, what we really wanted is to have a special sequence (K_n) giving rise to the h^n for any X, so we would like to find how to use this group operation for arbitrary X. This is achieved by considering the "loop space" ΩK of K, given by all basepoint preserving maps $S^1 \to K$.[58] Indeed, we then have the "adjoint relation" $\langle \Sigma X, K \rangle = \langle X, \Omega K \rangle$. This can be seen as follows: for any basepoint preserving map $f \colon \Sigma X \to K$, consider the maps obtained by restricting f to the images of $\{x\} \times I$ for varying x under the map $X \times I \to \Sigma X$; this maps X to a loop in K.
In particular, if $X = S^n$, we have that $\Sigma X = S^{n+1}$ so that $\pi_{n+1}(K) = \langle S^{n+1}, K \rangle = \langle S^n, \Omega K \rangle = \pi_n(\Omega K)$, which shows us that ΩK has all homotopy groups shifted down a dimension. As a consequence, we have that $\Omega K(G, n) = K(G, n-1)$.
We have then managed to put a group structure on $\langle X, \Omega K \rangle$, but we haven't yet made sure that these groups are abelian. One way of doing this is to instead consider $\langle X, \Omega^2 K \rangle$, and then by arguing similarly than for the commutativity of higher homotopy groups we can show that this is abelian (as it has the group structure of $\langle \Sigma^2 X, K \rangle$). Hence, to define h_n in terms of a sequence (K_n), we are led to the assumption that the K_n are all double loop

we need to consider $BU(n)$ and $BO(n)$, the cohomology of these spaces are generated by the Chern and Stiefel-Whitney classes, respectively, and pullback of them gives "characteristic classes" associated to the vector bundles. See [MS74] for a detailed account.

[57]Finite dimensional Lens spaces arise as follows: for integers p and q, choose q integers $(l_1, \ldots, l_q)$ coprime to p. Then the Lens space $L_p(l_1, \ldots, l_q)$ is defined to be the quotient space $S^{2q-1}/\mathbb{Z}_p$ given by considering $S^{2q-1} \subset \mathbb{C}^q$ and the action $\rho(z_1, \ldots, z_q) = \left(e^{\frac{2\pi i l_1}{p}} z_1, \ldots, e^{\frac{2\pi i l_q}{p}} z_q\right)$. For example, when $p = 2$, ρ is necessarily going to be the antipodal map, so that $L_2(l_1, \ldots, l_q) \cong \mathbb{RP}^{2q-1}$.

[58]This is made into a topological space using the compact-open topology: a subbasis for this topology is given by all maps whose image of a compact set is contained in an open set.

spaces. In fact, because we are only considering homotopy classes of maps, we just need that the K_n be double loop spaces up to homotopy. In this light, a sufficient condition is a homotopy equivalence $K_n \to \Omega K_{n+1}$: one of the reasons for this is that it guarantees that $K_n \to \Omega^2 K_{n+2}$ is a homotopy equivalence, hence that all K_n are homotopy equivalent to double loop spaces. In fact, the previous homotopy equivalence can actually be just a weak homotopy equivalence, which is also sufficient.[59]
This motivates the following definition:

Definition 4.20 An Ω-spectrum is a sequence of CW complexes (K_n) together with weak homotopy equivalences $K_n \to \Omega K_{n+1}$ for all n.

Theorem 4.21 If (K_n) is an Ω-spectrum then $h_n(X) = \langle X, K_n \rangle$ determines a reduced cohomology theory.[60]

Proof. Omitted, see [Hat02]. ❖

Corollary 4.22 For any abelian group G, the (reduced) cohomology theory determined by $K_n = K(G, n)$ is the usual (singular) cohomology with coefficients in G.

Proof. (Sketch) The previous theorem tells us that $h^n(X) = [X, K(G, n)]$ defines a reduced cohomology theory. We see that $h^n(S^k) = [S^k, K(G, n)] = \pi_k(K(G, n))$ which is G if $n = k$ and 0 otherwise, so that $h^n(S^k) \cong \widetilde{H}^n(S^k; G)$. This is enough to show that the two theories agree in the case of CW complexes, see [Hat02] for a proof.[61] ❖

Remark 4.23 The acclaimed Brown Representability Theorem shows that every reduced cohomology theory on CW complexes arises from this construction. This reveals an intimate connection between cohomology theories and loop spaces, indeed infinite loop spaces as $K_n = \Omega^k K_{n+k}$ for all $k \geqslant 1$. The infinite loopspaces $K(G, n)$ give rise to singular cohomology, but other spectra lead to different theories. For example, the infinite dimensional unitary and orthogonal groups, U and O, possess weak homotopy equivalences $U \to \Omega^2 U$ and $O \to \Omega^8 O$ as a consequence of the astonishing Bott Periodicity Theorem. These spectra give rise, respectively, to complex and real K-theory.

Group cohomology

The fact that classifying spaces are unique up to weak homotopy equivalence tells us that for a discrete group G, the cohomology of BG is well defined, as weakly homotopy

[59]By a Theorem of Milnor [Mil59], it is in fact the case that loop spaces of CW complexes are CW complexes themselves, so that it doesn't really matter if the homotopy equivalences are weak or not by Whitehead's Theorem 2.23, as long as we are considering X to be a CW complex.
[60]More precisely, the functors $X \mapsto h_n(X)$ form a reduced cohomology theory in the category of basepointed CW complexes and basepoint preserving continuous maps.
[61]The agreement of the two theories is stronger than described, as it takes the form of natural bijections between $h^n(X)$ and $\widetilde{H}^n(X; G)$; in a sense, the theories really are the same.

equivalent spaces have the same (co)homology.[62]
This allows us to consider the groups $H^n(BG)$ as a group cohomology of the group G.
There is also another construction that gives us the cohomology of a discrete group G. Start with the group ring $\mathbb{Z}[G]$, whose elements are finite formal linear combinations $g = \sum_{x \in G} n_x x$, where addition is the usual addition of linear combinations, and multiplication is given by

$$gh = \sum_{x \in G} \sum_{y \in G} n_x m_y xy = \sum_{z \in G} \left(\sum_{xy=z} n_x m_y \right) z$$

for $g = \sum_{x \in G} n_x x$ and $h = \sum_{y \in G} m_y y$.
We then consider a free resolution (or we could consider a projective resolution) of a $\mathbb{Z}[G]$ module M, using $\mathbb{Z}[G]$ modules:

$$\cdots \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$$

such a resolution always exists (see, for example, [AM94]).
Consider $\mathbb{Z}$ as a $\mathbb{Z}[G]$ module with trivial action. We can construct a free resolution of $\mathbb{Z}$ by considering our previous construction of EG and BG. We know that the cells of EG are all of the form $g \times \Delta^n \times (g_1, \dots, g_n)$, which we write as $g|g_1|\cdots|g_n|$, hence we see that we can realise F_n as $\coprod \mathbb{Z}[G]|g_1|\cdots|g_n|$ where the $(g_1, \dots, g_n)$ run over all elements of G^n with no coordinate being e. The boundary map is then given by

$$\partial|g_1|\cdots|g_n| = g_1|g_2|\cdots|g_n| + \sum_{i=1}^{n-1}(-1)^i|g_1|g_2|\cdots|g_{i-1}|g_i g_{i+1}|g_{i+2}|\cdots|g_n| + (-1)^n|g_1|\cdots|g_{n-1}|$$

and extended linearly, where we understand that if $g_i g_{i+1}$ is e we consider that summand to be 0; computation shows that this is indeed a valid boundary map.
We now apply the Hom functor to this resolution and take the cohomology, so that we get the groups $\mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z})$; this can in fact be done in more generality for any resolution.

Proposition 4.24 $\mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z}) = H^n(BG, \mathbb{Z})$

Proof. The previous free resolution of $\mathbb{Z}$ was given by the cellular chain complex of EG. Because $\mathrm{Hom}_{\mathbb{Z}[G]}(C_k(EG), \mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(C_k(BG), \mathbb{Z})$ (by our construction of EG and BG), this shows that the Ext functors agree with the cellular cohomology of BG (as the Ext functors are therefore taking the cohomology of the cellular cochain complex of BG). ❖

[62] We proved this for homotopy equivalence in Corollary 2.9. The fact that this also holds for weakly homotopy equivalent spaces is proved in [Hat02].

4.9 Poincaré Duality

From the wedge product

As we already know, the wedge product defines a map $H^k(M) \times H^l(M) \to H^{k+l}(M)$ by $([\omega],[\eta]) \mapsto [\omega \wedge \eta]$. In the case where $l = n-k$, this gives us a map to $H^n(M)$, which we know we can pair with $\mathbb{R}$ when M is compact and orientable. In more generality, we can consider compactly supported differential forms on any orientable manifold, and the wedge product gives a map $H^k(M) \times H_c^{n-k}(M) \to H_c^n(M) \cong \mathbb{R}$. This means that we can effectively view the wedge product as a map $\mathrm{PD}\colon H^k(M) \to (H_c^{n-k}(M))^*$ by $\mathrm{PD}\colon [\omega] \mapsto \left([\eta] \mapsto \int_M \omega \wedge \eta\right)$ called the Poincaré Dual map.

Theorem 4.25 For orientable manifolds without boundary M, the Poincaré Dual map $\mathrm{PD}_\wedge\colon H^k(M) \to (H_c^{n-k}(M))^*$ induced by the wedge product is an isomorphism for all k.

Proof. We will only prove the case when M has a finite acyclic cover (a cover where all open sets and their intersections are contractible (or empty)).
We will do this by induction on the number of open sets in a finite acyclic cover of M. If the cover has only one open set, the Poincaré dual map is clearly an isomorphism.
Now, suppose the theorem is true for every manifold which has a finite acyclic cover with at most c open sets, and say $\{U_1, \dots, U_c, V\}$ is a finite acyclic cover for M. Write $U = \cup_{i=1}^c U_i$, and consider the following diagram:

$$\begin{array}{ccccccccc}
H^{k-1}(U) \oplus H^{k-1}(V) & \longrightarrow & H^{k-1}(U \cap V) & \longrightarrow & H^k(M) & \longrightarrow & H^k(U) \oplus H^k(V) & \longrightarrow & H^k(U \cap V) \\
\downarrow{\scriptstyle \mathrm{PD}_\wedge \oplus \mathrm{PD}_\wedge} & & \downarrow{\scriptstyle \mathrm{PD}_\wedge} & & \downarrow{\scriptstyle \mathrm{PD}_\wedge} & & \downarrow{\scriptstyle \mathrm{PD}_\wedge \oplus \mathrm{PD}_\wedge} & & \downarrow{\scriptstyle \mathrm{PD}_\wedge} \\
\left(H_c^{n-k+1}(U) \oplus H_c^{n-k+1}(V)\right)^* & \longrightarrow & \left(H_c^{n-k+1}(U \cap V)\right)^* & \longrightarrow & \left(H_c^{n-k}(M)\right)^* & \longrightarrow & \left(H_c^{n-k}(U) \oplus H_c^{n-k}(V)\right)^* & \longrightarrow & \left(H_c^{n-k}(U \cap V)\right)^*
\end{array}$$

whose rows come from the Mayer-Vietoris sequences in de Rham cohomology and compactly supported cohomology; therefore the top one is exact, the bottom one is dualised but as all objects are vector spaces, it is also exact. The diagram commutes up to sign: Consider first the square

$$\begin{array}{ccc}
H^k(M) & \xrightarrow{k^* + l^*} & H^k(U) \oplus H^k(V) \\
\downarrow{\scriptstyle \mathrm{PD}_\wedge} & & \downarrow{\scriptstyle \mathrm{PD}_\wedge \oplus \mathrm{PD}_\wedge} \\
\left(H_c^{n-k}(M)\right)^* & \xrightarrow[(k_*)^* + (l_*)^*]{} & \left(H_c^{n-k}(U) \oplus H_c^{n-k}(V)\right)^*
\end{array}$$

and take $[\omega] \in H^k(M)$. We have that $(\mathrm{PD}_\wedge \oplus \mathrm{PD}_\wedge) \circ (k^* + l^*)(\omega)$ is given by

$$([\psi_1],[\psi_2]) \mapsto \int_U k^*(\omega) \wedge \psi_1 + \int_V l^*(\omega) \wedge \psi_2$$

On the other hand, $((k_*)^* + (l_*)^*) \circ \mathrm{PD}_\wedge(\omega)$ is given by

$$([\psi_1], [\psi_2]) \mapsto \int_M \omega \wedge k_*(\psi_1) + \int_M \omega \wedge l_*(\psi_2)$$

so that the square is indeed commutative as we can consider the second integrals as integrals over U and V respectively because $\mathrm{supp}(k_*(\psi_1)) \subset U$, $\mathrm{supp}(l_*(\psi_2)) \subset V$ and similarly for ω.
We can use an analoguous argument for the leftmost square, so we now only need to check the following square:

$$\begin{array}{ccc} H^{k-1}(U \cap V) & \xrightarrow{\delta} & H^k(M) \\ \downarrow{\scriptstyle \mathrm{PD}_\wedge} & & \downarrow{\scriptstyle \mathrm{PD}_\wedge} \\ \left(H_c^{n-k+1}(U \cap V)\right)^* & \xrightarrow{(\delta')^*} & \left(H_c^{n-k}(M)\right)^* \end{array}$$

where δ and δ' are the connecting homomorphisms given by the Zigzag Lemma. For $[\omega] \in H^{k-1}(U \cap V)$, $\delta\omega \in H^k(M)$ is given by the class of any form η such that $\eta|_U = -d(\rho_V \omega)$ and $\eta|_V = d(\rho_U \omega)$ where $\{\rho_U, \rho_V\}$ is a partition of unity subordinate to $\{U, V\}$.
For $[\psi] \in H_c^{n-k}(M)$, $\delta'\psi \in H_c^{n-k-1}(U \cap V)$ is given by the class of any form μ such that $-i_*(\mu) = d(\rho_U \psi)$ on U and $j_*(\mu) = d(\rho_V \omega)$ on V.
Now, for $[\omega] \in H^{k-1}(U \cap V)$ and $[\psi] \in H_c^{n-k}(M)$ we have that

$$\begin{aligned} (\mathrm{PD}_\wedge \circ \delta)([\omega])([\psi]) &= \int_M \delta\omega \wedge \psi \\ &= \int_{U \cap V} \delta\omega \wedge \psi \quad \text{as } \mathrm{supp}(\delta\omega) \subset U \cap V \\ &= \int_{U \cap V} d(\rho_U \omega) \wedge \psi \\ &= \int_{U \cap V} (d\rho_U) \wedge \omega \wedge \psi \quad \text{as } \omega \text{ is closed} \end{aligned}$$

and

$$\begin{aligned} ((\delta')^* \circ \mathrm{PD}_\wedge)([\omega])([\psi]) &= \int_{U \cap V} \omega \wedge \delta'\psi \\ &= \int_{U \cap V} \omega \wedge (-d(\rho_U \psi)) \\ &= -\int_{U \cap V} \omega \wedge d\rho_U \wedge \psi \quad \text{as } \psi \text{ is closed} \end{aligned}$$

so that the two integrals coincide up to sign, which is what we wanted.
Hence, by changing some of the vertical maps to their negatives we obtain a commutative diagram. All vertical maps are then isomorphisms by the induction hypothesis, except

possibly the middle one. By the Five Lemma, this shows that the middle arrow is an isomorphism too, which completes the induction and proves the theorem when M has a finite acyclic cover. It is possible to finish the argument by appealing to an "induction on open sets" result, which allows us to conclude that the Poincaré duality map is an isomorphism even when M does not have a finite acyclic cover, see [MT97]. ❖

In particular, if all cohomology groups of M are finite dimensional, then $H^k(M) \cong H_c^{n-k}(M)$ (in this case we say that M is of finite type).[63] If M is also compact, then we just have that $H^k(M) \cong H^{n-k}(M)$.
We might also want to consider the pairing $H_c^k(M) \to (H^{n-k}(M))^*$, but this is not necessarily an isomorphism: consider the infinite disjoint union $M = \coprod_{i=1}^{\infty} M_i$, we have that $H^k(M) = \prod_{i=1}^{\infty} H^k(M_i)$ but $H_c^k(M) = \bigoplus_{i=1}^{\infty} H_c^k(M_i)$. As explained in [BT84], the dual of a direct sum is a direct product, but the dual of a direct product is not necessarily a direct sum, so we have that $H^k(M) \cong (H_c^{n-k}(M))^*$ but not necessarily that $H_c^k(M) \to (H^{n-k}(M))^*$.

From oriented intersection theory

A version of Poincaré Duality also exists for homology using intersection:
We say that two submanifolds N_1 and N_2 of M intersect transversely if for all $p \in N_1 \cap N_2$, $T_pM = T_pN_1 \oplus T_pN_2$. The importance of this definition lies in the following proposition:

Proposition 4.26 Suppose N_1 and N_2 intersect transversely. Then $N_1 \cap N_2$ is a submanifold of M, with $\operatorname{codim}(N_1) + \operatorname{codim}(N_2) = \operatorname{codim}(N_1 \cap N_2)$.

Proof. Omitted, see [Hir94]. ❖

Now, consider two submanifolds N_1 and N_2 of a compact manifold M of complementary (co)dimension. We perturb then slightly so that they become transversal, and then compute the oriented intersection number, which is defined as follows:
For all $p \in N_1 \cap N_2$ (which is just a finite set of points by the above proposition and Bolzano-Weierstrass), the oriented intersection number at p is 1 if, given positively oriented bases $(v_1, \ldots, v_k)$, $(v_{k+1}, \ldots, v_{k+l})$ of N_1 and N_2, $(v_1, \ldots, v_k, \ldots, v_{k+l})$ is positively oriented, and -1 if it is negatively oriented. The oriented intersection number $\langle N_1, N_2 \rangle$ is then the sum of all these oriented intersection numbers over all points in the intersection. We would need to check that this is indeed well defined. It is known that transversality is "dense", as we can perturb the submanifolds an arbitrarily small amount and still get them to be transverse. It is also true that the oriented intersection number does not depend on which homotopies we have used to deform the manifolds. Proofs of both of

[63] These conditions are really necessary because, as [Spi99] notices, $H^1(\mathbb{R}^2 \setminus \mathbb{N}) \ncong H_c^1(\mathbb{R}^2 \setminus \mathbb{N})$: the first is in bijection with sequences of natural numbers, while the second is in bijection with sequences of natural numbers with only finitely many terms nonzero. The second is countably infinite, the first is uncountable.

these facts can be found in [Hir94].
By considering the fundamental classes associated to each (sub)manifold, this in fact gives us a map $H_k(M) \times H_{n-k}(M) \to \mathbb{Z}$. Hence we have a map $H_k(M) \to \mathrm{Hom}(H_{n-k}(M), \mathbb{Z})$, which is not always an isomorphism because of the existence of torsion elements, for example $\mathbb{RP}^3$ has homology $H_0 = \mathbb{Z}$, $H_1 = \mathbb{Z}_2$,$H_2 = 0$ and $H_3 = \mathbb{Z}$. We then see that $\mathbb{RP}^3$ is orientable ($H_3 \cong \mathbb{Z}$), it is also compact and connected, but $H_1 \ncong H_2$.
However we can get rid of torsion elements by considering coefficients in $\mathbb{Q}$, this then gives:

Theorem 4.27 For compact orientable manifolds without boundary, the map $\mathrm{PD}_\cap \colon H_k(M;\mathbb{Q}) \to \mathrm{Hom}(H_{n-k}(M;\mathbb{Q}),\mathbb{Q})$ induced by calculation of oriented intersection number is an isomorphism.

Proof. This will follow from the relation we will derived $\mathrm{PD}_\cap$ to $\mathrm{PD}_\wedge$. ❖

From the cap product

In the case of compact manifolds, Poincaré duality can also be made explicit by using the cap product $\frown \colon H_p(M;R) \times H^q(M;R) \to H_{p-q}(M;R)$ for $p \geqslant q$, given by $\sigma \frown \varphi = \varphi(\sigma|[v_0,\dots,v_q])\sigma|[v_q,\dots,v_p]$. One can check that this is indeed well defined in terms of (co)homology, and that the cap product is related to the cup product via $\psi(\sigma \frown \varphi) = (\varphi \smile \psi)(\sigma)$.
The Poincaré duality map for any R-orientable manifold is then given by $\mathrm{PD}_\frown(\psi) = [M] \frown \psi$ where $[M]$ is the fundamental class of M; see [Hat02] for details and proof that this is well defined.
We can relate Poincaré duality to integration along submanifolds: given an $(n-k)$-dimensional oriented submanifold N, integration over N defines a pairing $H_{n-k}(M) \times H_c^{n-k}(M) \to \mathbb{R}$; once again this is well defined by Stokes' Theorem. So we have an element in $(H_c^{n-k}(M))^*$ given by $[\omega] \mapsto \int_N \omega$, hence the first version of Poincaré duality (Theorem 4.25) gives us an isomorphism $(H_c^{n-k}(M))^* \cong H^k(M)$: there is a $[\eta] \in H^k(M)$ such that for all $[\omega] \in H_c^k(M)$, $\int_N \omega = \int_M \eta \wedge \omega$.
This then defines the Poincaré dual of a submanifold, as a map $\mathrm{PD}_S \colon H_{n-k}(M) \to H^k(M)$ (due to $\mathrm{PD}_\wedge$).
Now, given an oriented submanifold N of a compact oriented manifold M, let $\mathrm{PD}_S([N]) = \omega_N$. Then $\mathrm{PD}_\frown(\omega_N) = [M] \frown \omega_N = [N]$, so that PD_S determines $\mathrm{PD}_\frown$; this is just a restatement of the above consideration with $\int_N \omega = \int_M \eta \wedge \omega$.
We can also relate PD_S to $\mathrm{PD}_\cap$: for k and $n-k$ dimensional submanifolds N_1 and N_2, we have that $\int_M \mathrm{PD}_S([N_1]) \wedge \mathrm{PD}_S([N_2]) = \langle N_1, N_2 \rangle$, showing finally how the intersection product and Poincaré duality coming from the wedge product are related. The proof of this fact is quite complicated, so we refer to [BT84] for a detailed exposition.

Applications

A useful application of Poincaré duality is the computation of the Euler characteristic of some manifolds:

Proposition 4.28 Let M be an odd-dimensional compact manifold (without boundary). Then $\chi(M) = 0$.

Proof. If M is orientable, Poincaré duality tells us that $\text{rank}(H_k(M)) = \text{rank}(H_{n-k}(M))$. As n is odd, the alternating sum $\sum_{k=0}^{n}(-1)^k\text{rank}(H_k(M))$ cancels in pairs, and we are done.
Now, suppose M is not orientable. The Poincaré duality using the cap product applies for any coefficient ring R when M is R-orientable, so in particular it applies for $R = \mathbb{Z}_2$. This shows that $\dim(H_k(M;\mathbb{Z}_2)) = \dim(H_{n-k}(M;\mathbb{Z}_2))$, as vector spaces over $\mathbb{Z}_2$, and hence that $\sum_{k=0}^{n}(-1)^k\dim(H_k(M;\mathbb{Z}_2))$, similarly to above. We now want to relate $H_k(M;\mathbb{Z}_2)$ to $H_k(M;\mathbb{Z}) = H_k(M)$.
From the Universal Coefficient Theorem for cohomology (slightly modified for vector spaces over $\mathbb{Z}_2$ instead of abelian groups), we have the following exact sequence:

$$0 \to \text{Ext}_{\mathbb{Z}_2}(H_{k-1}(M;\mathbb{Z}_2),\mathbb{Z}_2) \to H^k(M;\mathbb{Z}_2) \to \text{Hom}_{\mathbb{Z}_2}(H_k(M;\mathbb{Z}_2),\mathbb{Z}_2) \to 0$$

where Ext is 0 as can be seen from the free resolution of $\mathbb{Z}_2$ vector spaces $0 \to H_{k-1}(M;\mathbb{Z}_2) \to H_{k-1}(M;\mathbb{Z}_2) \to 0$. As all (co)homology groups of a compact manifold are finitely generated, we get that $H_k(M;\mathbb{Z}_2) \cong H^k(M;\mathbb{Z}_2)$. Using the Universal Coefficient Theorem again, we have the exact sequence:

$$0 \to \text{Ext}(H_{k-1}(M),\mathbb{Z}_2) \to H^k(M;\mathbb{Z}_2) \to \text{Hom}(H_k(M),\mathbb{Z}_2) \to 0$$

Now, consider $H_k(M)$ decomposed using the classification theorem for finitely generated abelian groups, starting with $k = 0$. In this case, each factor of $\mathbb{Z}$ in $H_k(M)$ contributes a factor of $\mathbb{Z}_2$ in $H^k(M;\mathbb{Z}_2)$, as $\text{Hom}(\mathbb{Z},\mathbb{Z}_2) \cong \mathbb{Z}_2$ and Hom commutes with direct sums. Similarly, a factor of $\mathbb{Z}_{2m}$ contributes a $\mathbb{Z}_2$ factor to $H^k(M;\mathbb{Z}_2)$ as $\text{Hom}(\mathbb{Z}_{2m},\mathbb{Z}_2) \cong \mathbb{Z}_2$, but it now also contributes to $H^{k+1}(M;\mathbb{Z}_2)$ as Ext measures the failure of the map $H^k(M;\mathbb{Z}_2) \to \text{Hom}(H_k(M),\mathbb{Z}_2)$ to be injective. $\text{Ext}(\mathbb{Z}_{2m},\mathbb{Z}_2) \cong \mathbb{Z}_2$, so it also contributes a factor of $\mathbb{Z}_2$, but to $H^{k+1}(M;\mathbb{Z}_2)$; and again, this discussion of contributing factors is well defined as Ext commutes with direct sums, see Proposition 4.13 for a justification of the properties of Ext. We also note that adding a factor of $\mathbb{Z}$ does not change Ext as $\text{Ext}(\mathbb{Z},\mathbb{Z}_2) \cong 0$. And a factor of $\mathbb{Z}_{2m+1}$ contributes nothing, as $\text{Hom}(\mathbb{Z}_{2m+1},\mathbb{Z}_2) \cong 0$, $\text{Ext}(\mathbb{Z}_{2m+1},\mathbb{Z}_2) \cong 0$. We can use these facts, increase k gradually, and this gives us $\dim(H_k(M);\mathbb{Z}_2)$ in terms of $\text{rank}(H_k(M))$ for all k.
Now, looking at $\chi(M) = \sum_{k=0}^{n}(-1)^k\text{rank}(H_k(M))$, shows us that for each k, if we change $\text{rank}(H_k(M))$ into $\dim(H_k(M);\mathbb{Z}_2)$, $\chi(M)$ doesn't change, as what contributes to the

rank of the first also contributes the same amount to the dimension of the second, and the factors which don't contribute to the rank of the first either cancel out (for $\mathbb{Z}_{2m}$) or don't change anything ($\mathbb{Z}_{2m+1}$). Hence $\chi(M) = \sum_{k=0}^{n}(-1)^k \dim(H^k(M, \mathbb{Z}_2)) = 0$. ❖

Another application is in relation to the oriented intersection numbers.
Consider a link (K_1, K_2) of two components in $\mathbb{R}^3$, and define the quantity $\mathrm{L}(K_1, K_2)$ of the link diagram by looking at all crossings where K_1 goes over K_2. At each crossing, look at the basis of the plane of the diagram, given by, say, a tangent vector in the positive direction of K_1 and then a tangent vector in the positive direction of K_2 (where the orientations come from the orientation of S^1). Then assign to each crossing the value $+1$ if the basis is positively oriented respective to the standard orientation of $\mathbb{R}^2$, and -1 otherwise. The sum of all these values is $\mathrm{L}(K_1, K_2)$.[64]

Proposition 4.29 $L(K_1, K_2)$ is the linking number $\ell(K_1, K_2)$ of K_1 and K_2 (provided we choose the correct orientations of $S^1 \times S^1$ and S^2).

Proof. Project K_2 flat on a plane (it does not matter that it can cross itself by being a nontrivial knot), and K_1 too except at crossings with K_1, where it leaves the plane in a small neighbourhood of the crossing, say along a small half-circle in a perpendicular plane. Then consider the map $q(u, v)\colon S^1 \times S^1 \to S^2$ given in the definition of the linking number. Each crossing where K_1 goes over K_2 gives us a point in $S^1 \times S^1$ that maps to the north pole of S^2, and these are the only such points. We can easily see that any point on S^2 is a regular value of q, and by looking at orientations we see that counting preimages (with signs) does indeed give us $L(K_1, K_2)$, at least up to sign. ❖

Remark 4.30 This shows that we can count the linking number by counting crossings with signs, and that this does not depend on which diagram we choose to represent the link.

Now, the linking number can be computed as follows. Put K_1 and K_2 as in the proof of the previous proposition, and realise K_2 as the boundary of some singular 2-chain c that lies entirely above (or on) the plane, this can be done as $H_1(\mathbb{R}^3) \cong 0$. Then we compute the oriented intersection number of K_1 and c. We see that if K_1 leaves the plane in small enough neighbourhoods, the intersections of K_1 and c are in bijection with the crossings of K_1 and K_2 where K_1 is above. We can consider which orientations we get, and we see that the oriented intersection number at each crossing is indeed what we want, so that $L(K_1, K_2) = \langle K_1, c\rangle$.
This definition extends to more general manifolds; we can repeat the exact same construction with M_1 and M_2, of respective dimensions p and q, as submanifolds of S^{p+q+1}, and get a linking number. But this also allows us to complete our treatment of Poincaré duality,

[64]Notice that we only considered crossings where K_1 goes over K_2. But as both knots are closed, we would get the same number (up to sign) if we considered crossings of K_2 over K_1, as for each crossing of K_1 over K_2 there must be a corresponding crossing of K_2 over K_1.

coming from the intersection product. We had a pairing $fH_k(M) \times fH_{n-k}(M) \to \mathbb{Z}$ from the free parts of $H_k(M)$ and $H_{n-k}(M)$ to the integers by computing the oriented intersection number, and this gave rise to an isomorphism $fH_k(M) \cong \mathrm{Hom}(fH_{n-k}(M), \mathbb{Z})$. But we can now consider the torsion parts: take representing elements $[c_1] \in tH_k(M)$ and $[c_2] \in tH_{n-k-1}(M)$. Because $[c_1]$ has a finite order, say p, we can represent pc_1 as ∂c for some singular $k+1$-chain c. The pairing is then given by $([c_1], [c_2]) \mapsto \frac{\langle c, c_2 \rangle}{p}$; this gives us a pairing $tH_k(M) \times tH_{n-k-1}(M) \to \mathbb{Q}/\mathbb{Z}$, where we have quotiented by $\mathbb{Z}$ as we could've represented any nonzero multiple of pc_1 instead of pc_1. This pairing induces an isomorphism $tH_k(M) \cong \mathrm{Hom}(tH_{n-k-1}(M), \mathbb{Q}/\mathbb{Z})$, called the torsion linking form; this name comes from the above discussion of this construction in relation with the linking number of links.

References

[AM94] Alejandro Adem and James Milgram, *Cohomology Of Finite Groups*, Springer, 1994.

[Arm83] Mark A. Armstrong, *Basic Topology*, Springer, 1983.

[BT84] Raoul Bott and Loring Tu, *Differential Forms in Algebraic Topology*, Springer, 1984.

[Hat02] Allen Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.

[Hir94] Morris W. Hirsch, *Differential Topology*, Springer, 1994.

[Lee00] John M. Lee, *Introduction To Smooth Manifolds*, Springer, 2000.

[Mac98] Saunders MacLane, *Categories For The Working Mathematician*, Second ed., Springer, 1998.

[Mil59] John W. Milnor, *On Spaces Having the Homotopy Type of a CW Complex*, Transactions Of The American Mathematical Society **90** (1959), pp 272–280.

[Mon06] David Mond, *Cohomology, Curvature, Connections And Characteristic Classes Lecture Notes*, 2006.

[Mor01] Shigeyuki Morita, *Geometry Of Differential Forms*, AMS, 2001.

[MS74] John W. Milnor and James D. Stasheff, *Characteristic Classes*, Princeton University Press, 1974.

[MT97] Ib H. Madsen and Jorgen Tornehave, *From Calculus To Cohomology*, Cambridge University Press, 1997.

[Pra94] Viktor V. Prasolov, *Elements Of Homology Theory*, AMS, 1994.

[Spi99] Michael Spivak, *A Comprehensive Introduction To Differential Geometry*, Third ed., vol. 1, Publish or Perish, 1999.

[Vic94] James W. Vick, *Homology Theory : An Introduction To Algebraic Topology*, Second ed., Academic Press, 1994.

[Wei94] Charles A. Weibel, *An Introduction To Homological Algebra*, Cambridge University Press, 1994.