Hilbert Spaces

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Introduction

Functional analysis is an important field of study in mathematics that builds on the ideas founded by a first linear algebra course. In particular, we can consider certain sets of functions, such as the set of continuous functions on a closed interval, as vector spaces and explore their properties with this abstract framework. Hilbert spaces offer more structure than vector spaces, and therefore makes their study easier.

This essay will look at Hilbert spaces and other related spaces, such as Banach spaces, and explore some of the results that can be obtained from their study.

1 Introducing Hilbert Spaces

1.1 Basic definitions

We will begin with some definitions for norms, inner products and their associated spaces.

Definition 1.1. *A* norm *on a complex vector space* V *is a function*

$$\|\cdot\| : V \to \mathbb{R}$$

such that, for all $x, y \in V$*, and for all* $\lambda \in \mathbb{C}$*, we have:*

(i) $\|x\| \geq 0$

(ii) $\|x\| = 0 \iff x = 0.$

(iii) $\|\lambda x\| = |\lambda| \, \|x\|.$

(iv) $\|x + y\| \leq \|x\| + \|y\|.$

A normed vector space $(V, \|\cdot\|)$ *is a complex vector space* V *paired with a norm on* V*. (from [1, p5])*

Note. This definition can be adapted for use when V is a real vector space. In this case, $|\lambda|$ is the usual absolute value of $\lambda \in \mathbb{R}$.

Definition 1.2. *An* inner product *on a complex vector space* V *is a function*

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$$

such that, for all $x, y, z \in V$*, and for all* $\lambda \in \mathbb{C}$*, we have:*

(i) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

(ii) $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$

(iii) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

(iv) $\langle x, x \rangle \geq 0$ *with* $\langle x, x \rangle = 0 \iff x = 0$

An inner product space $(V, \langle \cdot, \cdot \rangle)$ *is a complex vector space* V *paired with an inner product on* V*. (from [3, p6])*

Note. In the case that V is a real vector space, we ignore the complex conjugate sign in part (i) since the complex conjugate of $x \in \mathbb{R}$ is just x.

Here is an example of a familiar space that is an inner product space:

Example. $\mathbb{R}^n$ becomes an inner product space when paired with the function

$$\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n$$

which is the usual dot product. It is easy to verify that it satisfies all the required properties of an inner product.

This proposition shows that the space of continuous complex-valued functions over the interval $[0, 1]$ is also an inner product space. We will now call this space $C[0, 1]$.

Proposition 1.1. *The function*

$$\langle f, g \rangle = \int_0^1 f(t)\overline{g(t)}dt$$

defines a inner product on $C[0, 1]$*, with pointwise addition and scalar multiplication.*

Proof. We will prove that the function satisfies all four required properties from Definition 1.2. In the proof, $f, g, h \in C[0, 1]$,

(i) Using the properties of complex integrals, we see that

$$\overline{\langle g, f \rangle} = \overline{\int_0^1 g(t)\overline{f(t)}dt} = \int_0^1 \overline{g(t)\overline{f(t)}}dt = \int_0^1 \overline{g(t)}f(t)dt = \langle f, g \rangle$$

(ii)

$$\langle \lambda f, g \rangle = \int_0^1 [\lambda f(t)]\overline{g(t)}dt = \lambda \int_0^1 f(t)\overline{g(t)}dt = \lambda \langle f, g \rangle$$

(iii)

$$\begin{aligned} \langle f + g, h \rangle &= \int_0^1 [f(t) + g(t)]\overline{h(t)}dt \\ &= \int_0^1 f(t)\overline{h(t)}dt + \int_0^1 g(t)\overline{h(t)}dt = \langle f, h \rangle + \langle g, h \rangle \end{aligned}$$

(iv) Let $f(t) = u(t) + iv(t)$. Then

$$\langle f, f \rangle = \int_0^1 f(t)\overline{f(t)}dt = \int_0^1 [u(t) + iv(t)]\overline{u(t) + iv(t)} = \int_0^1 u(t)^2 + v(t)^2 dt$$

Let $f(t) \neq 0$. This means that either $u(t)$, $v(t)$ is non-zero. Withlout loss of generality, let $u(t) > 0$ and $v(t) \geq 0$. Thus, $u(t)^2 > 0$, $v(t)^2 \geq 0$. This means that the integral of $u(t)^2 + v(t)^2$ is going to be non-negative, as required. To have the inner product to be zero, since the integrand is greater than or equal to zero for all t, you require $u(t)$ and $v(t)$ to both be zero for all t, and hence the whole function must be zero. ☐

Note. The proposition and proof can be easily adapted to show that the result holds for the more general space of functions $C[a, b]$.

1.2 Results about norms and inner products

This section mostly follows [1, pp6-9].

Definition 1.3. *Let V be an inner product space, and $v \in V$. We define the* norm *of v to be*

$$\|v\| := \sqrt{\langle v, v\rangle}$$

To check that this norm satisfies the criteria from Definition 1.1, we will need to utilise the following lemma and theorem:

Lemma 1.2. *Let V be an inner product space, and $u, v, w \in V$. Then:*

(i) $\langle u, \lambda v\rangle = \overline{\lambda}\,\langle u, v\rangle$.

(ii) $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$.

Proof. (i) $\langle u, \lambda v\rangle = \overline{\langle \lambda v, u\rangle} = \overline{\lambda} \cdot \overline{\langle v, u\rangle} = \overline{\lambda}\,\langle u, v\rangle$

(ii) $\langle u, v + w\rangle = \overline{\langle v + w, u\rangle} = \overline{\langle v, u\rangle} + \overline{\langle w, u\rangle} = \langle u, v\rangle + \langle u, w\rangle$ ☐

Theorem 1.3. (Cauchy-Schwartz inequality) *Let $x, y \in V$, where V is a inner product space. Then*

$$\langle x, y\rangle \leq \sqrt{\langle x, x\rangle}\sqrt{\langle y, y\rangle}$$

(from [3, p7])

Proposition 1.4. $\|v\| := \sqrt{\langle v, v\rangle}$ *is a norm on V as defined in Definition 1.1.*

Proof. For this proof, $v, w \in V$.

(i) By the properties of inner products, we have $\langle v, v\rangle \geq 0$. Thus

$$\|v\| = \sqrt{\langle v, v\rangle} \geq 0$$

(ii) Again, by the properties of inner products, we have $\langle v, v\rangle = 0 \iff v = 0$. Therefore, since $\sqrt{x} = 0 \iff x = 0$, we have that

$$\|v\| = \sqrt{\langle v, v\rangle} = 0 \iff v = 0$$

(iii) Using Lemma 1.2

$$\|\lambda v\| = \sqrt{\langle \lambda v, \lambda v\rangle} = \sqrt{\lambda\,\langle v, \lambda v\rangle} = \sqrt{\lambda\bar{\lambda}\,\langle v, v\rangle} = \sqrt{|\lambda|^2\,\langle v, v\rangle} = |\lambda|\,\|v\|$$

(iv) Using Lemma 1.2 and the Cauchy-Schwartz inequality, we have

$$\begin{aligned}
\|x + y\|^2 &= \langle x + y, x + y\rangle \\
&= \langle x, x\rangle + \langle y, y\rangle + \langle x, y\rangle + \langle y, x\rangle \\
&\leq \langle x, x\rangle + \langle y, y\rangle + 2\sqrt{\langle x, x\rangle}\sqrt{\langle y, y\rangle} \\
&= (\sqrt{\langle x, x\rangle} + \sqrt{\langle y, y\rangle})^2 \\
&= (\|x\| + \|y\|)^2
\end{aligned}$$

Since both sides of the equation are positive, taking square roots gives the required result. ☐

The consequence of being able to define a norm from an inner product is the following corollary:

Corollary 1.5. *All inner product spaces are normed vector spaces.*

The converse is not always true; a given normed vector space is not necessarily an inner product space. However, there is a way to tell if a given normed vector space has an inner product associated with it.

Proposition 1.6. (*Parallelogram law) If V is a normed vector space, and for all $x, y \in V$ we have*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

then V is an inner product space.

If this property does hold, you can use the the following identity to calculate what the inner product is.

Proposition 1.7. *For any x, y in an inner product space, we have*

$$\langle x, y\rangle = \frac{1}{4}\sum_{n=0}^{3} i^n \|x + i^n y\|^2$$

1.3 Banach and Hilbert spaces

Definition 1.4. *from [1, p23]*

- *A* Banach space *is a normed vector space that is complete with respect to the metric induced by the norm.*
- *A* Hilbert space *is an inner product space that is complete with respect to the metric induced by the norm.*

From this definition, we can see that all Hilbert spaces are Banach spaces. However, the converse is not true. Take ℓ^∞, the set of all bounded complex sequences. With component-wise addition and scalar multiplication, this is a normed vector space, and is complete with respect to the supremum norm

$$\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$$

It is therefore a Banach space. However, it does not satisfy the parallelogram law. Take $x = (2, 1, 0, 0, \ldots)$ and $y = (1, 2, 0, 0, \ldots)$. Then

$$\|x+y\|_\infty = 3$$

$$\|x-y\|_\infty = 1$$

$$\|x\|_\infty = \|y\|_\infty = 2$$

So

$$\|x+y\|_\infty^2 + \|x-y\|_\infty^2 = 10 \neq 16 = 2\|x\|_\infty^2 + 2\|y\|_\infty^2$$

It is therefore not a inner product space, and thus not a Hilbert space.

1.4 The space L^2

Let us consider the space $C[-1,1]$ with the inner product

$$\langle f, g\rangle = \int_0^1 f(t)\overline{g(t)}dt$$

This has the associated norm

$$\sqrt{\int_0^1 |f(t)|^2 dt}$$

This inner product space is not complete; consider the continuous function

$$f_n(x) = \begin{cases} 1, & x \in [-1,0] \\ 1 - nx, & x \in (0, 1/n] \\ 0, & x \in (1/n, 1] \end{cases}$$

This converges to

$$f(x) = \begin{cases} 1, & x \in [-1,0] \\ 0, & x \in (0,1] \end{cases}$$

which is clearly not continuous.

We could try to enlarge[1] the class of functions that we consider with this inner product. However, the space of Riemannian integrable functions on the interval $[-1,1]$ that are square-integrable, that is, satisfy

$$\int_{-1}^1 |f(t)|^2 dt < \infty$$

is still not complete. Henri Lesbegue developed a new notion of integration. Lesbegue's method of integration enabled functions to be integrated that previously could not, such as

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Definition 1.5. *We define $L^2[a,b]$ to be the completion of C[a,b] with respect to $\|\cdot\|_2$. (from [3, p88])*

The space $L^2[a,b]$ contains all the functions that were integrable by Riemann's method, and also includes functions that are only integrable by Lesbegue's method. All of the functions contained in this space are square-integrable. It is also an inner product space, with an inner product function similar[2] to the one outlined for the space $C[a,b]$. Therefore, it is a Hilbert space.

2 Exploring Hilbert spaces

2.1 Separability

We will start with a general notion of separability that can be applied to any topological space.

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[1] This paragraph is paraphrased from [1, p24].

[2] The inner product function can be found in Saxe's text, [3, p76].

Definition 2.1. *A topological space T is* separable *if it contains a countable dense subset. (from [2, p57])*

Equivalently, T is separable if there exists a sequence $(x_n)_{n=1}^{\infty}$ with all its terms in T such that every non-empty open subset of T contains at least one term of the sequence.

Example.

- Using the Euclidean metric, $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$. Therefore, $\mathbb{R}$ is separable.
- Similarly, let $\mathbb{Q}^n := \{(r_1, \ldots, r_n) : r_i \in \mathbb{Q}\}$. Then $\mathbb{Q}^n$ is countable, and dense in $\mathbb{R}^n$. Thus, $\mathbb{R}^n$ is separable.

We can also show that the normed vector space $C[0, 1]$ is separable, using Weierstrass' Approximation Theorem:

Theorem 2.1. *(Weierstrass' Approximation Theorem) For all $\varepsilon > 0$ and for all $f \in C[a, b]$, there exists a polynomial $p \in \mathbb{R}[x]$ such that $\|f - p\|_\infty < \varepsilon$. (from [3, p136])*

To show that $C[0, 1]$ is separable, we require the following proposition:

Proposition 2.2. *Let (X, d) be a metric space, and let $A \subseteq B \subseteq X$. If A is dense in B (with respect to the subspace topology on B), and if B is dense in X, then A is dense in X. (from [4])*

We know that $C[0, 1]$ is a metric space with respect to the supremum norm. Weierstrass' approximation theorem states that the polynomials $\mathbb{R}[x]$ are dense in $C[0, 1]$. It can be shown that $\mathbb{Q}[x]$ is a dense subset of $\mathbb{R}[x]$. Combining this information gives us the following corollary:

Corollary 2.3. *The set $\mathbb{Q}[x]$ is dense in $C[0, 1]$ with respect to the supremum norm.*

Since the set $\mathbb{Q}[x]$ is countable we now have that $C[0, 1]$ is indeed separable.

2.2 Separability in normed vector spaces

In this section, we look at separability from the point of view of Hansen's text, [2, pp56-57].

Recall that for a vector space V, if there exists a finite set $\{x_1, \ldots, x_n\}$ contained in V such that, for all $v \in V$, we have

$$v = a_1 x_1 + \cdots + a_n x_n$$

where $a_i \in \mathbb{C}$ for $1 \leq i \leq n$, we call the set a (finite) basis for V. In this case, we know that V has finite dimension n.

We have a similar notion for a basis for an infinite dimensional normed vector space.

Definition 2.2. *Let V be a normed vector space. If there exists a sequence $(x_n) \in V$ such that every $v \in V$ admits a unique decomposition as the sum of a convergent series*

$$v = \sum_{n=1}^{\infty} a_n x_n \qquad a_n \in \mathbb{C} \quad \forall\, n \in \mathbb{N}$$

we call the sequence a Schauder basis *(from [2, p56].*

Note. The order of the terms in the decomposition is important, as the sum may not be unconditionally convergent.

With these definitions for bases, we can consider a different idea of separability for normed vector spaces.

Definition 2.3. *If a normed vector space V admits a finite basis, or a Schauder basis, we say V is* separable by a basis.

This kind of separability is stronger than the general version, as the following theorem states:

Theorem 2.4. *If a normed vector space V is a separable by a basis, then it is separable.*

2.3 Orthogonality

Definition 2.4. *For an inner product space V, x and y in V are* orthogonal *if $\langle x, y\rangle = 0$.*

Using this definition of orthogonality, we can create a generalised version of Pythagoras' theorem.

Theorem 2.5. *Let x, y be elements of an inner product space V. If x, y are orthogonal, then*

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2$$

(from [2, p54])

Proof. Since x, y are orthogonal, then $\langle x, y\rangle = \langle y, x\rangle = 0$. Therefore,

$$\begin{aligned}\|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, x\rangle \\ &= \langle x, x\rangle + \langle y, y\rangle \\ &= \|x\|^2 + \|y\|^2\end{aligned}$$

□

Definition 2.5. *(from [1, p31])*

- *A family $(e_a)_{a\in A}$ in $V\backslash\{0\}$ where A is an index set is called an* orthogonal system *if $\langle e_i, e_j\rangle = 0$ when $i \neq j$ for all $i, j, \in A$.*
- *An orthogonal system is called an* orthonormal system *if $\|e_a\| = 1$ for all $a \in A$.*
- *An orthonormal system is called an* orthonormal sequence *if the family (e_a) can be indexed by $\mathbb{N}$.*

Note. In the definition for an orthonormal system, the condition that $\|e_a\| = 1$ for all $a \in A$ implies that $\langle e_i, e_j\rangle = \delta_{ij}$ where δ_{ij} for all $i, j \in A$ is the Kronecker delta.

With this definition, we can extend Theorem 2.5 as follows: if $x_1, \ldots, x_n$ is an orthogonal system in an inner product space V, then

$$\left\|\sum_{k=1}^{n} x_k\right\|^2 = \sum_{k=1}^{n} \|x_k\|^2$$

2.4 Complete orthonormal sequences

In Fourier analysis, we can decompose certain functions into sums of sines and cosines. In general, we can try to decompose elements of a Hilbert space into sums of the form

$$\sum_{k=1}^{\infty} c_k x_k$$

where $(x_k)_{k=1}^{\infty}$ is an orthonormal sequence. When is this decomposition possible?

Definition 2.6. *An orthonormal sequence* (e_n) *in a Hilbert space* H *is* complete *if the only element of* H *that is orthogonal to every terms of the sequence is the zero vector. (from [1, p36])*

Note. A complete orthonormal sequence is often called an *orthonormal basis.*

Note. This definition of "complete" is unrelated to complete metric spaces.

We find that the decomposition into a sum is only possible if you use a orthonormal basis in the sums. So, for which Hilbert spaces is it possible to decompose elements in such a way? Here, we use the notion of separability discussed earlier.

Proposition 2.6. *A Hilbert space* H *has a orthonormal basis iff* H *is separable. (from [5])*

We would now like to know what the coefficients c_k are in the sum decomposition.

Theorem 2.7. *Suppose that* $x = \sum_{k=1}^{\infty} c_k x_k$ *for an complete orthonormal sequence* $(x_k)_{k=1}^{\infty}$ *in an inner product space* V*. Then* $c_k = \langle x, x_k \rangle$ *for each* k*.*

Proof. Let $s_n := \sum_{k=1}^{n} c_k x_n$. So

$$\lim_{n\to\infty} \|s_n - x\| = 0$$

Fix m and let $n > m$. Then, using the Cauchy-Schwartz inequality,

$$\begin{aligned}\langle s_n, x_m \rangle - \langle x, x_m \rangle &= \langle s_n - x, x_m \rangle \\ &\le \|s_n - x\| \cdot \|x_m\| \\ &= \|s_n - x\|\end{aligned}$$

noting that x_m is a term from an orthonormal sequence. Therefore

$$\lim_{n\to\infty} \langle s_n, x_m \rangle = \langle x, x_m \rangle$$

Now consider $\langle s_n, x_m \rangle$.

$$\begin{aligned}\langle s_n, x_m \rangle = \left\langle \sum_{k=1}^{n} c_k x_k, x_m \right\rangle &= \sum_{k=1}^{n} \langle c_k x_k, x_m \rangle \\ &= \sum_{k=1}^{n} c_k \langle x_k, x_m \rangle = \sum_{k=1}^{n} c_k \delta_{km} \\ &= c_m\end{aligned}$$

c_m is a constant not dependent on n, so c_m remains unchanged as $n \to \infty$. Therefore

$$\langle x, x_m \rangle = \lim_{n\to\infty} \langle s_n, x_m \rangle = c_m$$

as required. □

Definition 2.7. *Let $(x_k)_{k=1}^{\infty}$ be an orthonormal sequence in an inner product space V, and let f be an element in V. We define*

$$\sum_{k=1}^{\infty} \langle x, x_k \rangle \, x_k$$

to be the Fourier series of f *with respect to $(x_k)_{k=1}^{\infty}$, and $\langle x, x_k \rangle$ to be the* Fourier coefficients of x *with respect to $(x_k)_{k=1}^{\infty}$. (from [3, p32])*

These are named after Joseph Fourier, whose work on the heat equation inspired the study into these sums.

2.5 Results on Fourier series

So far, we have only defined these sums formally; we do not know if they converge yet. The following two results[3] assure us that these sums do indeed converge, and thus the decomposition makes sense.

Theorem 2.8. *(Bessel's inequality) Let $(x_n)_{n=1}^{\infty}$ be an orthonormal sequence in a Hilbert space H. Then, for all $x \in H$, we have*

$$\sum_{n=1}^{\infty} | \langle x, x_n \rangle |^2 \leq \|x\|^2$$

Proposition 2.9. *Let $(x_n)_{n=1}^{\infty}$ be an orthonormal sequence in a Hilbert space H, and let $(a_n)_{n=1}^{\infty}$ be a sequence of complex numbers. Then*

$$\sum_{n=1}^{\infty} a_n x_n \text{ is convergent in } H \iff \sum_{n=1}^{\infty} |a_n|^2 \text{ is convergent in } \mathbb{C}$$

Bessel's inequality tells us that the sum

$$\sum_{n=1}^{\infty} | \langle x, x_n \rangle |^2$$

is convergent, since that sum is less than or equal to some finite number. Proposition 2.9 then tells us that the sum

$$\sum_{n=1}^{\infty} \langle x, x_n \rangle \, x_n$$

is also convergent in H, which is the Fourier series for the arbitrary element $x \in H$.

To conclude, we will find out under what circumstances Bessel's inequality has equality.

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[3] These are from [2, pp59,60]

Theorem 2.10. *(Parseval's Theorem) Suppose* $(x_k)_{k=1}^{\infty}$ *is an orthonormal sequence in a Hilbert space* H*. Then*

$$\sum_{k=1}^{\infty} |\left\langle x, x_k \right\rangle|^2 = \|x\|^2 \qquad \forall\ x \in H$$

if and only if $(x_k)_{k=1}^{\infty}$ *is an orthonormal basis.*

Proof. This proof is based on the one given by Hansen, [2, pp64,65].

$\Rightarrow$) Let $x \in H$. Since $\sum_{n=1}^{\infty} |\left\langle x, x_n \right\rangle|^2$ is convergent in H then

$$\sum_{n=1}^{\infty} \left\langle x, x_n \right\rangle x_n := y$$

is also convergent and defines a vector in H. Now, consider $\left\langle y, x_k \right\rangle$ for some $k \in \mathbb{N}$. Define $c_n := \left\langle x, x_n \right\rangle$. Then

$$\begin{aligned}
\left\langle y, x_k \right\rangle &= \left\langle \sum_{n=1}^{\infty} c_n x_n, c_k \right\rangle \\
&= \left\langle c_1 x_1, x_k \right\rangle + \left\langle c_2 x_2, x_k \right\rangle + \cdots \\
&= \sum_{n=1}^{\infty} \left\langle c_n x_n, x_k \right\rangle \\
&= \sum_{n=1}^{\infty} c_n \left\langle x_n, x_k \right\rangle \\
&= \sum_{n=1}^{\infty} c_n \delta_{nk} = c_k = \left\langle x, x_k \right\rangle
\end{aligned}$$

Now consider $\left\langle x - y, x_k \right\rangle$ for some $k \in \mathbb{N}$.

$$\left\langle x - y, x_k \right\rangle = \left\langle x, x_k \right\rangle - \left\langle y, x_k \right\rangle = \left\langle x, x_k \right\rangle - \left\langle x, x_k \right\rangle = 0$$

We therefore have that

$$\|x - y\| = \sum_{k=1}^{\infty} |\left\langle x - y, x_k \right\rangle|^2 = \sum_{k=1}^{\infty} 0 = 0$$

By the properties of norms, we have that $x = y$, and thus

$$x = \sum_{n=1}^{\infty} \left\langle x, x_n \right\rangle x_n$$

Since the x is arbitrarily chosen, we have that the sequence $(x_n)_{n=1}^{\infty}$ is in fact an orthonormal basis.

$\Leftarrow$) Since $(x_k)_{k=1}^{\infty}$ is an orthonormal basis, we can write any element $x \in H$ as

$$x = \sum_{k=1}^{\infty} c_k x_k$$

where $c_k := \langle x, x_k \rangle$ as before. Then

$$\begin{aligned}
\|x\|^2 = \langle x, x \rangle &= \left\langle \sum_{n=1}^{\infty} c_n x_n, \sum_{k=1}^{\infty} c_k, x_k \right\rangle \\
&= \left\langle c_1 x_1, \sum_{k=1}^{\infty} c_k x_k \right\rangle + \left\langle c_2 x_2, \sum_{k=1}^{\infty} c_k, x_k \right\rangle + \cdots \\
&= \sum_{n=1}^{\infty} \left\langle c_n x_n, \sum_{k=1}^{\infty} c_k x_k \right\rangle \\
&= \sum_{n=1}^{\infty} \langle c_n x_n, c_1 x_1 \rangle + \langle c_n x_n, c_2 x_2 \rangle + \cdots \\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \langle c_n x_n, c_k x_k \rangle \\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} c_n \overline{c_k} \langle x_n, x_k \rangle \\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} c_n \overline{c_k} \delta_{nk} \\
&= \sum_{n=1}^{\infty} |c_n|^2
\end{aligned}$$

Therefore

$$\|x\|^2 = \sum_{n=1}^{\infty} |c_n|^2 = \sum_{n=1}^{\infty} | \langle x, x_k \rangle |^2$$

as required. ☐

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