Curvature

Contents

1 Differential geometry of curves

1.1 Frenet-Serret formulas

We would like to start by considering curves $\gamma\colon [a,b] \to \mathbb{R}^n$. In this case, we say a curve is regular of order k if γ is of class C^k and $(\gamma'(t), \gamma''(t), \ldots, \gamma^{(k)}(t))$ are linearly independent for all $t \in [a,b]$.

Given a plane curve (say, regular of order 2), we can consider the tangent vector $\gamma'(t)$, and the unit tangent vector $T(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|}$. If γ is parametrised by arclength, we have that $\|\gamma'(t)\| = 1$ for all t, from now on consider that this is the case by choosing an arclength parametrisation for γ. We can then define the normal vector $\frac{dT}{dt}$ and the unit normal vector N similarly. We then have that, at each point, $\frac{dT}{dt} = \kappa N$, and we call κ the curvature of γ (at t).

We know that this definition of curvature coincides with the geometric understanding that we should define the curvature of a circle of radius r as $\frac{1}{r}$. This is indeed the case as given $\gamma(t) = (r\cos(\frac{t}{r}), r\sin(\frac{t}{r}))$, $\|\gamma''(t)\| = \| -\frac{1}{r}(\cos(\frac{t}{r}))^2 - \frac{1}{r}(\sin(\frac{t}{r}))^2\| = \frac{1}{r}$, where the factor of $\frac{1}{r}$ ensures that γ is parametrised by arclength. Then we can easily see that the curvature of any curve is just the curvature of the osculating circle to the curve at that point, that is, the circle that agrees with the curve for first and second-order behaviours of γ.

Now, for $\gamma\colon [a,b] \to \mathbb{R}^3$, we can also consider the binormal $B = T \times N$ where $\times$ is the vector product. This then gives us an orthonormal basis for $\mathbb{R}^3$ at each point $\gamma(t)$ for $t \in [a,b]$. As $\langle B, B\rangle = 1$, we have that $\frac{d}{dt}\langle B, B\rangle = 0$ so $\langle \frac{d}{dt}B, B\rangle + \langle B, \frac{d}{dt}B\rangle = 0$ so that $\frac{d}{dt}B$ is a linear combination of T and N only. But $\langle B, T\rangle = 0$ so $\frac{d}{dt}\langle B, T\rangle = \langle \frac{d}{dt}B, T\rangle + \langle B, \frac{d}{dt}T\rangle = 0$, so $\langle \frac{d}{dt}B, T\rangle = -\langle B, \frac{d}{dt}T\rangle = -\langle B, N\rangle = 0$ so B is in fact just a multiple of N at each point $\gamma(t)$, so that $B = -\tau N$, where τ is called the torsion of γ at t. We can then see τ as measuring how far γ is from being planar, as for a planar curve, B is a constant vector normal to the plane in which the curve is contained.

We can now express the relationship between T,N,B and their derivatives:

Theorem 1.1 (Frenet-Serret formulas)

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

Proof. Consider the matrix $Q = \begin{pmatrix} T \\ N \\ B \end{pmatrix}$. The rows of Q are orthonormal vectors, so that Q is an orthogonal matrix. Writing $A = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}$, we want to show $\frac{dQ}{dt} = AQ$ so $A = \frac{dQ}{dt}Q^{-1} = \frac{dQ}{dt}Q^T$. But we have already shown this equality for the first and last columns, so it suffices to show that $\frac{dQ}{dt}Q^T$ is skew-symmetric. But $QQ^T = I$ so $\frac{dQ}{dt}Q^T + Q\frac{dQ^T}{dt} = 0$, hence $Q\frac{dQ^T}{dt} = -\left(Q\frac{dQ^T}{dt}\right)^T$, as required. ❖

The importance of the curvature and torsion is given by the following theorem:

Theorem 1.2 Fundamental Theorem Of Curves
Two curves in $\mathbb{R}^3$ are congruent (via Euclidean motions) if and only if their arclength parametrisations have the same curvature and torsion at every point.

Proof. Omitted; this theorem is mainly a result about ordinary differential equations: we can easily picture a curve in $\mathbb{R}^3$ with the corresponding osculating circle and torsion, there is only one way we can "integrate" these invariants to get the curve back. This then only depends on the initial data we chose, which is a starting point and a "velocity" vector for the curve at the starting point, corresponding to the Euclidean motion needed to move one curve onto another. ❖

This theory of curves extends naturally to curves $\gamma \colon [a, b] \to \mathbb{R}^n$, where we consider the orthonormal frame obtained by Gram-Schmidt orthonormalisation of the frame $(T, T', \ldots, T^{(n)})$.[1]

1.2 Hopf's Umlaufsatz

Given a differentiable curve in the plane, we can consider its total curvature as $\int_\gamma \kappa ds$. If the curve is not differentiable but just piecewise differentiable, we can generalise this integral to mean $\sum_i \int_{\gamma_i} \kappa ds + \sum_j \delta_j$ where δ is the exterior angle corresponding to a non-differentiable point which is just the (oriented) angle by which the tangent vector to the curve is moved at that point. We can also write δ as $\pi - \alpha$ where α is the corresponding interior angle at that point.

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[1] In this case, we can write the generalised Frenet-Serret formulas and we end up with $k - 1$ distinct curvatures, the m-th curvature measuring how much the curve fails to be contained in a m dimensional (affine) subspace of $\mathbb{R}^n$.

Theorem 1.3 (Hopf's Umlaufsatz)
Let γ be a simple closed piecewise-differentiable positively oriented curve in the plane. Then $\int_\gamma \kappa ds = 2\pi$.

Proof. Let $\gamma\colon [0,L] \to \mathbb{R}^2$ be such a curve. Without loss of generality, we can assume that γ is actually differentiable. Consider the triangle $\Delta = \{(x,y) : 0 \leqslant x \leqslant y \leqslant L\}$, and define $\varphi\colon \Delta \to S^1$ as follows:

$$\begin{aligned}
\varphi(x,y) &= \frac{\gamma(y)-\gamma(x)}{\|\gamma(y)-\gamma(x)\|} \quad \text{for } x<y \text{ and } (x,y) \neq (0,L) \\
\varphi(x,x) &= \gamma'(x) \\
\varphi(0,L) &= -\gamma'(0)
\end{aligned}$$

This definition makes φ continuous. The idea is then to deform the tangent $\varphi(x,x)$ (which corresponds to keeping $x=y$ and moving them along) to moving y first and then x. As the total curvature $\int_\gamma \kappa ds$ is necessarily a multiple of 2π (it is the total change in the angle of the tangent vector), this deformation must preserve the integral $\int_\gamma \kappa ds$, seeing as it has to stay at a multiple of 2π and can't jump between different multiples as φ is continuous. But if we consider moving along y first and then along x, we get two curves $\gamma_1\colon [0,L] \to \mathbb{R}^2$ and $\gamma_2\colon [L,2L] \to \mathbb{R}^2$. In each case the total curvature is just the angle by which the vector from $\gamma(0)$ to $\gamma_1(t)$ (or $\gamma(0)$ and $\gamma_2(t)$) changes, so we get that $\int_\gamma \kappa ds = \pi + \pi = 2\pi$. ❖

2 Differential geometry of surfaces

2.1 The second fundamental form and Gauss's Theorema Egregium

Following the idea of measuring curvature of curves with circles, we can consider measuring the curvature of surfaces with their relation to the sphere. Given a surface $M \subset \mathbb{R}^3$, one can define the normal map (also called Gauss map) $\nu \colon M \to S^2$ that sends the point $p \in M$ to the unit normal vector to M at p, which corresponds to a point on S^2.

This map also gives rise to the map $d\nu$, called the Weingarten map, given by considering the action of the derivative of ν on the tangent vectors of M at x. Seeing as $d\nu$ essentially maps the tangent vectors to a point $x \in M$ to tangent vectors to the point $\nu(x) \in S^2$, and that the two tangent planes are naturally parallel as they both have the same normal vector, we can consider $d\nu$ as a map from tangent vectors to $x \in M$ to themselves, given by applying the derivative and identifying the two tangent planes at x and $\nu(x)$.

The second fundamental form II is then defined to be $\mathrm{II}(X, Y) = -\langle d\nu(X), Y\rangle$ where $\langle , \rangle$ is the standard inner product on $\mathbb{R}^3$, which is also called the first fundamental form $\mathrm{I}(X, Y) = \langle X, Y\rangle$.[2]

Proposition 2.1 The second fundamental form II is symmetric, so that $\mathrm{II}(X, Y) = \mathrm{II}(Y, X)$.

Proof. Following [Spi99a], consider, locally, M as the image of a map $f \colon \mathbb{R}^2 \to \mathbb{R}^3$, and write $N = \nu \circ f$. This gives a parametrisation of M, $f(u, v)$. Then $d\nu\left(\frac{\partial f}{\partial v}\right) = \frac{\partial}{\partial v}(\nu \circ f) = \frac{\partial N}{\partial v}$.

We know that $\left\langle N, \frac{\partial f}{\partial u}\right\rangle = 0$ as N is normal to M and $\frac{\partial f}{\partial u}$ is tangent to M at the same

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[2] We can in fact define the third fundamental form and so on by $\mathrm{III}(X, Y) = \big\langle (d\nu)^2(X), Y\big\rangle$ (where ^{2} means applying the map twice), but in fact all the higher fundamental forms are readily expressible in terms of I and II as by the Cayley Hamilton Theorem, $d\nu$ satisfies $(-d\nu)^2 - \mathrm{tr}(-d\nu)(-d\nu) + \det(-d\nu) = (-d\nu)^2 - 2H(-d\nu) + K\mathrm{id} = 0$; H and K are defined just after the next remark.

point. We can now differentiate this identity to get:

$$\begin{aligned}\left\langle N, \frac{\partial^2 f}{\partial u \partial v}\right\rangle &= -\left\langle \frac{\partial N}{\partial v}, \frac{\partial f}{\partial u}\right\rangle \\ &= \left\langle d\nu\left(\frac{\partial f}{\partial v}\right), \frac{\partial f}{\partial u}\right\rangle \\ &= \mathrm{II}\left(\frac{\partial f}{\partial v}, \frac{\partial f}{\partial u}\right)\end{aligned}$$

Doing the same thing with u and v reversed, we get that

$$\left\langle N, \frac{\partial^2 f}{\partial v \partial u}\right\rangle = \mathrm{II}\left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}\right)$$

so using equality of second mixed partial derivatives, we get that II is symmetric, because it is symmetric on the basis $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}$. ❖

Remark 2.2 The symmetry of II implies that $\langle d\nu(X), Y\rangle = \langle X, d\nu(Y)\rangle$. We then know (via the Spectral Theorem) that $d\nu$ has an orthonormal basis of real eigenvectors, so we can consider the eigenvalues of $d\nu$ in the orthogonal directions corresponding to the orthonormal basis of eigenvectors.

This allows us to define the Gaussian curvature K of M at x to be the product of the eigenvalues of $d\nu$, and the mean curvature H to be half the sum of the eigenvalues.

Proposition 2.3 For any two linearly independent tangent vectors X_1, X_2 we have that $K = \frac{\det(\mathrm{II}(X_1,X_2))}{\det(\mathrm{I}(X_1,X_2))}$.

Proof. Suppose X_1 and X_2 are orthonormal. We know that $\mathrm{II}(X_1, X_2) = \langle -d\nu(X_1), X_2\rangle$, so that

$$K = \det(d\nu) = \begin{vmatrix} -\mathrm{II}(X_1, X_1) & -\mathrm{II}(X_2, X_1) \\ -\mathrm{II}(X_1, X_2) & -\mathrm{II}(X_2, X_2)\end{vmatrix} = \det(\mathrm{II})$$

If X_1 and X_2 are not orthonormal, write $X_i = \sum_j a_{ij} Y_j$ with orthonormal Y_1, Y_2, and replace Y_1, Y_2 by X_1, X_2. This multiplies both $\det(\mathrm{II}(Y_1, Y_2))$ and $\det(\mathrm{I}(Y_1, Y_2))$ by $\det(a_{ij})$ so the formula still holds, as it holds for Y_1, Y_2. ❖

Proposition 2.4 Given a curve $\gamma\colon [-a, a] \to M$, parametrised by arclength, with tangent vector $\gamma'(0) = X$, we have that $\langle \gamma''(X), \nu\rangle = \mathrm{II}(X, X)$ at the point $p = \gamma(0)$.

Proof. As explained in [Spi99a], at a point p we have that $\frac{d\nu(\gamma)}{dt}(p) = d\nu(X)$ and $\langle \gamma'(t), \nu(p)\rangle = 0$. Therefore $\langle \gamma''(0), \nu(p)\rangle + \langle \gamma'(0), d\nu(X)\rangle = 0$ so $\langle \gamma''(0), \nu(p)\rangle = -\langle \gamma'(0), d\nu(X)\rangle = -\langle X, d\nu(X)\rangle = \mathrm{II}(X, X)$. ❖

This proposition explains two aspects of the second fundamental form we have defined: first of all, this means that $\mathrm{II}(X, X)$ is the usual (signed) curvature of the curve obtained by cutting the surface M along the plane through p containing X and $\nu(p)$. This gives a curve c_X with $\langle (c_X)''(0), \nu(p)\rangle = \langle \gamma''(0), \nu(p)\rangle$ but with $(c_X)''(0)$ perpendicular to $\nu(p)$, and $\|(c_X)''(0)\|$ is precisely the usual curvature (but unsigned). Secondly, this means that we can consider the Gaussian curvature K as the product of the principal curvatures, which are the extremal curvatures of the curves cut out by orthogonal planes through p containing $\nu(p)$ by Remark 2.2.

When considering maps f from a surface M to another surface N, the natural idea is then to ask that they conserve I, so that $\mathrm{I}(X, Y) = \mathrm{I}(df(X), df(Y))$, such maps are called local isometries. These maps correspond to our usual intuition for deforming surfaces into others, for example, there is a local isometry between the plane (a sheet of paper) and the cylinder (and we can indeed roll the paper easily into a cylinder) but no such local isometry between the cylinder and the torus (as we can see that we have to crease the cylinder to form a torus). Notice that we can write the first and second fundamental forms as matrices: given a basis of the tangent plane to M at p, we can write $\mathrm{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$ and $\mathrm{II} = \begin{pmatrix} L & N \\ N & M \end{pmatrix}$ (these matrices are functions of p on the whole surface).

Theorem 2.5 (Brioschi Formula)

$$K = \frac{1}{(EG - F^2)^2}\left(\begin{vmatrix} -\frac{1}{2}\frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 F}{\partial x \partial y} - \frac{1}{2}\frac{\partial^2 G}{\partial x^2} & \frac{1}{2}\frac{\partial E}{\partial x} & \frac{\partial F}{\partial x} - \frac{1}{2}\frac{\partial E}{\partial y} \\ \frac{\partial F}{\partial y} - \frac{1}{2}\frac{\partial G}{\partial x} & E & F \\ \frac{1}{2}\frac{\partial G}{\partial y} & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}\frac{\partial E}{\partial y} & \frac{1}{2}\frac{\partial G}{\partial x} \\ \frac{1}{2}\frac{\partial E}{\partial y} & E & F \\ \frac{1}{2}\frac{\partial G}{\partial x} & F & G \end{vmatrix} \right)$$

Proof. Omitted, the proof of this is just a large computation. Refer to [Spi99a] for details. ❖

The Brioschi formula then tells us that the Gaussian curvature is an intrinsic invariant of our surface, as it depends only on our first fundamental form. From this follows Gauss's remarkable theorem:

Corollary 2.6 (Theorema Egregium)

Any two locally isometric surfaces have the same Gaussian curvature.

Proof. This is immediate from the Brioschi formula which tells us that the Gaussian curvature can be defined purely in terms of the first fundamental form. ❖

2.2 Gauss-Bonnet Theorem

The Gauss-Bonnet provides a surprising relation between the local structure of the surface (here, its Gaussian curvature) and its global properties (its Euler characteristic):

Theorem 2.7 (Gauss-Bonnet)
Let $M \subset \mathbb{R}^3$ be a compact orientable surface without boundary. Then

$$\int_M K dA = 2\pi\chi(M)$$

Proof. Omitted. See, for example, Spivak [Spi75a]. It is a consequence of the generalised Gauss-Bonnet Theorem which we shall prove later on. ❖

Suppose we now consider a polygonal shape S on the surface, diffeomorphic to a subset of $\mathbb{R}^2$ (so that $\chi(S) = 1$). In the plane, we had that $\int_\gamma \kappa ds = 2\pi$ by Hopf's Umlaufsatz. We no longer want to be considering the usual curvature of a curve however, as it "goes out" of the surface. Hence we define the geodesic curvature k_g of a curve on M to be the projection of the curvature vector of the curve onto the tangent plane to the surface at that point. We can now "patch together" Hopf's Umlaufsatz and the Gauss-Bonnet Theorem.

Theorem 2.8 (Local Gauss-Bonnet)
Let $M \subset \mathbb{R}^3$ be a orientable surface and S a polygonal subset of M as described above. Then

$$\int_S K dA + \int_{\partial S} k_g ds = 2\pi$$

Proof. Omitted, see Spivak [Spi75a]. ❖

Note that we once again interpret $\int_{\partial S} k_g ds$ to mean $\sum_i \int_{\partial S} k_g ds + \sum_j \delta_j$ where δ are the exterior angles as described before.
We can state this theorem in more generality if we don't only consider polygons which are homeomorphic to $\mathbb{R}^2$, and we get:

Theorem 2.9 (Gauss-Bonnet for surfaces with boundary)
Let $M \subset \mathbb{R}^3$ be a compact orientable surface. Then

$$\int_M K dA + \int_{\partial M} k_g ds = 2\pi\chi(M)$$

Proof. This follows, for example, from constructing a triangulation of M. We know that for each triangle we have that $\int_S K dA = -\int_{\partial T} k_g ds + 2\pi$, so adding up the integrals we are led to consider $\int_{\partial M} k_g ds = \sum_{i=1}^{F} \int_{\partial S_i} k_g ds + \sum_{i=1}^{F} (\delta_{1i} + \delta_{2i} + \delta_{3i})$, and we get that $\sum_{i=1}^{F} \int_{\partial S_i} k_g ds = \int_{\partial M} k_g ds$ as each edge of the triangulation not on the boundary of M appears twice in opposite orientations, $\sum_{i=1}^{F} (\delta_{1i} + \delta_{2i} + \delta_{3i}) = -\sum_{i=1}^{F} (\alpha_{1i} + \alpha_{2i} + \alpha_{3i}) + \sum_{i=1}^{F} 3\pi = -2\pi V + 3\pi F = -2\pi V + 2\pi E$ as $3F = 2E$ and $\sum_{i=1}^{F} 2\pi = 2\pi F$, so by adding together and using $\chi(M) = V - E + F$, we get that $\int_M K dA + \int_{\partial M} k_g ds = 2\pi\chi(M)$. ❖

2.2.1 Angular defects

The Gauss-Bonnet sheds a new light on the theorems of classical geometry relating to the sum of the angles of a triangle. In the plane, both the Gaussian and the geodesic curvatures corresponding to a triangle are 0, so the Gauss-Bonnet Theorem correctly tells us that $\alpha_1 + \alpha_2 + \alpha_3 = \pi$. On a sphere, if we consider geodesic triangles T (formed by arcs of great circles), we see that their geodesic curvature is 0 as the curvature vector is normal to the tangent plane. We know that the sphere of radius R has Gaussian curvature $\frac{1}{R^2}$ everywhere, so the Gauss-Bonnet Theorem says that $\int_T \frac{1}{R^2} dA + \pi = \sum_{i=1}^{3} \alpha_i$. But $\int_{S^2} \frac{1}{R^2} dA = \frac{\text{Area}(T)}{R^2}$ so $\sum_{i=1}^{3} \alpha_i = \pi + \frac{\text{Area}(T)}{R^2}$. For the pseudosphere of "radius" R (which provides a model for hyperbolic geometry), the Gaussian curvature is $\frac{-1}{R^2}$, and similarly we get that $\sum_{i=1}^{3} \alpha_i = \pi - \frac{\text{Area}(T)}{R^2}$ for a geodesic triangle.

The Gauss-Bonnet Theorem can also be applied to polyhedrons, in which we consider the curvature to be concentrated at the vertices and 0 elsewhere, and we have to do similar considerations to define the integral as we did for piecewise-smooth paths. The correct notion of curvature in this case is then the angular defect at a vertex, which is $D = 2\pi - \sum_i \alpha_i$ where the α_i are the angles between the edges of each face at that vertex, this is indeed a sensible definition as if the polyhedron is flat around one vertex, we see that $D = 0$, and the more the polyhedron is curved, the larger (in absolute value) D will be. This then leads to the following theorem:

Theorem 2.10 (Descartes)

Let M be a polyhedron (in $\mathbb{R}^3$) with vertices $v_1, \ldots, v_n$ and corresponding defects $D_1, \ldots, D_n$. Then:

$$\sum_{i=1}^{n} D_i = 2\pi\chi(M)$$

2.3 The Darboux frame

We can also apply the ideas of the Frenet-Serret frame to get a frame on a surface $M \subset \mathbb{R}^3$. In this case, around a point $p \in M$, we require that M does not have an umbilical point (a point in which the principal curvatures are equal. Then, given a curve $\gamma\colon [-1,1] \to M$, parametrised by arclength and regular of class C^2, with $\gamma(0) = p$, we can define the tangent vector $T(t) = \gamma'(t)$, the unit normal to the surface at $\gamma(t)$: $n(t)$, and the tangent-normal $b(t) = n(t) \times T(t)$.
We can then try to relate this frame to the moving frame of γ given by the same T, and with $N(t) = \frac{T'(t)}{\|T'(t)\|}$, $B(t) = T(t) \times N(t)$. As both frames are orthonormal, (n, b) must be a rotation of (N, B), so we can write $\begin{pmatrix} T \\ n \\ b \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$.
Using the Frenet-Serret formulas, we then have:

$$\begin{aligned}
\begin{pmatrix} T' \\ n' \\ b' \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin(\alpha)\frac{d\alpha}{dt} & -\cos(\alpha)\frac{d\alpha}{dt} \\ 0 & \cos(\alpha)\frac{d\alpha}{dt} & -\sin(\alpha)\frac{d\alpha}{dt} \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix}^{-1} \begin{pmatrix} T \\ n \\ b \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin(\alpha)\frac{d\alpha}{dt} & -\cos(\alpha)\frac{d\alpha}{dt} \\ 0 & \cos(\alpha)\frac{d\alpha}{dt} & -\sin(\alpha)\frac{d\alpha}{dt} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix}^{-1} \begin{pmatrix} T \\ n \\ b \end{pmatrix} \\
&= \begin{pmatrix} 0 & \kappa\cos(\alpha) & \kappa\sin(\alpha) \\ -\kappa\cos(\alpha) & 0 & \tau \\ -\kappa\sin(\alpha) & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ n \\ b \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{d\alpha}{dt} \\ 0 & \frac{d\alpha}{dt} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{pmatrix} \begin{pmatrix} T \\ n \\ b \end{pmatrix}
\end{aligned}$$

where κ_g is the geodesic curvature, κ_n is the normal curvature (such that $\kappa_g + \kappa_n = \kappa$) and τ_g is the geodesic torsion defined by $\tau_g = \tau + \frac{d\varphi}{dt}$ where φ is the angle from the osculating plane to the tangent plane, which is precisely $-\alpha$.

3 Curvature in higher dimensions

3.1 Concepts of differential geometry

To generalise our study of curves and surfaces to higher dimensions, notice that curves are locally modelled on $\mathbb{R}$ and surfaces look locally like $\mathbb{R}^2$. The following definition makes this intuition precise:

Definition 3.1 A manifold M of dimension m is a Hausdorff[3] topological space X together with an open cover of X, $\{U_i\}$, such that each U_i is homeomorphic to an open subset of $\mathbb{R}^m$.[4] Manifolds are also often required to be second-countable.[5]

The additional conditions of a manifold being Hausdorff or second-countable avoids pathological examples: the Hausdorff conditions ensures uniqueness of limits, and second countability avoids spaces which are "too large", such as the long line.[6] Second-countable Hausdorff spaces are normal[7] and admit partitions of unity,[8] which are properties that usual curves and surfaces definitely have and we would expect to generalise. Some examples of manifolds: $\mathbb{R}^n$, with charts being the identity, the circle S^1, which has an open cover consisting of the circle minus one point, and the circle minus another point, and the charts are the stereographic projection from the removed point. The n-sphere S^n is an n dimensional manifold and is constructed similarly.

Hence, with these definitions, 1-manifolds are curves and 2-manifolds are surfaces. Note that the converse is not true: non-simple curves are not manifolds as they have singular points at points of intersection, and the neighborhood of such points does not resemble

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[3] A Hausdorff topological space is a topological space in which any two distinct points can be separated by neighbourhoods, so that we can find two open sets, one containing each point, that have empty intersection.

[4] These homeomorphisms $\varphi\colon U \to \mathbb{R}^n$ are called charts, and their collection, covering the whole manifold, is called an atlas.

[5] A second countable topological space has a countable basis for the topology, so that there exists a countable set of open sets which allows us to write any open set as an union of elements of this countable set.

[6] The long line can be constructed as follows: take the set $\omega_1 \times [0,1)$ (where ω_1 is the first uncountable ordinal) with the lexicographical order, and give it the order topology. This is just putting together an uncountable number of copies of the interval $[0,1)$ with the expected topology. Now take the disjoint union of that space with the same space but with the order reversed and $(0,0)$ removed, and give an order on that by saying that elements in the reversed space are always smaller than in the not reversed space. This space with the order topology corresponds to an uncountable number of intervals $[0,1)$ (or $(0,1]$ if in the reversed part) put together.

[7] A topological space is normal iff any two disjoint closed sets can be separated by neighborhoods.

[8] A partition of unity of a manifold M is a set of C^k functions $p_i\colon M \to [0,1]$ such that, for every point $x \in M$, there exists a neighborhood $x \in U \subset M$ such that finitely many functions are nonzero on that neighborhood and that $\sum_i p_i = 1$. One can also consider partitions of unity "subordinate" to an open cover $\{U_i\}$, which is a partition of unity satisfying support$(p_i) \subset U_i$ for all i.

$\mathbb{R}$ (rather, it resembles two copies of $\mathbb{R}$ that intersect). This notion of singular point is made more precise in algebraic geometry, and in many circumstances it is possible to say that a manifold is a non-singular algebraic variety.[9]

For the moment, we have only considered gluing together the pieces of the manifolds by homeomorphisms, but we would like to generalise the methods of vector calculus that we have on $\mathbb{R}^n$, which involves notions of differentiability. We might be tempted to consider only charts of class C^k. However to consider differentiable manifolds we rather need to consider how the charts are put together in the atlas. For any two charts φ_a and φ_b, define the transition map $\varphi_{ab} = \varphi_a \circ (\varphi_b)^{-1}$ as a map $\varphi_{ab} \colon \varphi_b(U_a \cap U_b) \to \varphi_a(U_a \cap U_b)$. A differentiable manifold of class C^k is then a manifold where all transitions maps are of class C^k. A differentiable manifold of class C^∞ is also called a smooth manifold. From now on and unless explicitly stated otherwise, all manifolds are assumed to be smooth, which avoids technical considerations of differentiability.

3.1.1 The tangent bundle

The full structure of curves and surfaces has not yet been generalised - for the moment, these manifolds lack the notion of tangent vectors that were useful for the Frenet frame for example. Tangent vectors are essentially one-dimensional as they can be defined as tangents to given curves at a point - this allows one definition of the tangent space of a manifold:

Definition 3.2 The tangent space T_xM of a manifold M at a point x is the set of all curves $\gamma \colon \mathbb{R} \to M$ with $\gamma(0) = x$ under the equivalence relation $\gamma_1 \sim \gamma_2$ if given a chart φ, the usual derivatives $\frac{\partial(\varphi \circ \gamma_1)}{\partial t}$ and $\frac{\partial(\varphi \circ \gamma_2)}{\partial t}$ are equal for $t = 0$. The tangent bundle TM is defined as the disjoint union $\coprod_{x \in M} T_xM$ of all tangent spaces.

One can also consider tangent vectors as objects that differentiate functions: in $\mathbb{R}^n$, we can see a tangent vector X as acting on a function $f \colon M \to \mathbb{R}$ by giving the (directional) derivative of f in the X direction, denoted $X(f) = df(X)$. The product rule of derivatives leads us to consider derivations: a derivation D is a linear map $D \colon C^\infty(M) \to \mathbb{R}$ such that $D(fg) = Df(g) + fD(g)$. We can see that the definition of tangent vectors to curves allows us to consider $D(f) = \frac{\partial(f \circ \gamma)}{\partial t}$, which is indeed a derivation as partial derivatives satisfy the product rule. This leads us to an alternative definition of the tangent space, as the set of derivations at x:

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[9] In algebraic geometry, we consider algebraic sets as the locus of points such that a polynomial defined on affine space vanishes, and algebraic varieties are then the irreducible algebraic sets, where irreducible means irreducible in the Zariski topology, in which the closed sets are taken to be the algebraic sets. Non-singular varieties are then varieties which have no singular points.

Proposition 3.3 *T_xM is the vector space of derivations $D\colon C^\infty(M) \to \mathbb{R}$.*

Proof. The above remarks show that any curve induces a derivation, and any derivation, written locally around a point as $X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$, can be considered as a tangent to a curve simply by constructing a curve γ by $\gamma(t) = \varphi^{-1}(ta_1, \ldots, ta_n)$. ❖

This proposition makes apparent the vector space structure of T_xM which isn't totally clear from the first definition. Given a chart φ, we write $\frac{\partial}{\partial x_i}$ for the derivation corresponding to the partial derivative along the i-th coordinate of the chart. Hence for a function $f\colon M \to \mathbb{R}$ and a tangent vector $X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$, we can write $X(f) = \sum_{i=1}^n a_i \frac{\partial f}{\partial x_i}$.

Proposition 3.4 *TM is a manifold.*

Proof. Omitted. See, for example, [Mor01]. ❖

Given a tangent vector $X \in T_pM$ and a map $f\colon M \to N$, f should transform X according to its first-order behaviour. More precisely, f induces a natural map $f_*\colon T_pM \to T_{f(p)}N$ called the pushforward of f by $f_*(\gamma'(0)) = (f(\gamma))'(0)$.[10] This shows that f_* acts on tangent vectors by "scaling" by its differential, as in the usual case of $\mathbb{R}^n$.

We can also consider the dual space to T_xM, called the cotangent space T_x^*M, and the cotangent bundle[11] $T^*M = \coprod_{x\in M} T_x^*M$. Given a (local) basis of T_xM $\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}\right)$, we write the dual basis as $(dx_1, \ldots, dx_m)$, so we have $dx_i\left(\frac{\partial}{\partial x_j}\right) = \delta_i^j$.

This knowledge about how f acts on tangent vectors allows us to formulate many convenient properties such a map can possess. We say that a map f is an immersion if f_* is everywhere injective, and f is a submersion if f is everywhere surjective. The basic examples of such maps are the standard immersion $i\colon R^n \to R^{n+m}$, $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0, \ldots, 0)$ and the standard submersion $s\colon R^{n+m} \to R^n$, $(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) \mapsto (x_1, \ldots, x_n)$. A related concept is that of regular values: for a function $f\colon M \to N$, a point $x \in M$ is a regular point if $f_*\colon T_xM \to T_{f(x)}N$ is surjective, and a point $y \in N$ is called a regular value if all points in $f^{-1}(y)$ are regular points. The interest of such functions lies in the following:

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[10] Considering the tangent vectors as derivations, we have the equivalent statement $f_*(X_p)(g) = X_p(g(f))$.

[11] There is another equivalent definition of the cotangent space: Consider the set of functions around p. This forms a local ring R, that is a ring with an unique maximal ideal, the ideal $\mathcal{I}$ of functions which vanish at p. The cotangent space is then defined as the the vector space $\mathcal{I}/\mathcal{I}^2$ over the field $R/\mathcal{I} \cong \mathbb{R}$. This shows that T_p^*M corresponds to functions which vanish at p modulo second-order behaviour, which is precisely what (co)tangent vectors measure.

Theorem 3.5 (Preimage Theorem)
Suppose $f\colon M \to N$ is a smooth map, and $y \in Y$ a regular value of f. Then $X = f^{-1}(y)$ is a submanifold of M, with $\dim(X) = \dim(M) - \dim(N)$.

Proof. Omitted, see [Lee00]. ❖

This theorem allows us to check easily if many spaces are manifolds; for example, the n-sphere can be described as $f^{-1}(0)$ with $f(x_0, \ldots, x_n) = 1 - \sum_{i=0}^{n} x_i^2$, and this does satisfy the conditions of the theorem.

3.1.2 Fiber bundles and vector bundles

Notice that the tangent and cotangent bundles look locally (in a neighborhood U of $p \in M$) like a product space $U \times F$. The definition of fiber bundle captures this notion:

Definition 3.6 A fiber bundle $\xi = (\pi, E, M, F)$ is a continuous surjection $\pi\colon E \to M$ such that around each point $x \in M$, there exists an open neighborhood U that makes the fiber $\pi^{-1}(U)$ locally a product $U \times F$.[12] The projection π and the projection onto the first factor p_1 of $U \times F$ have to agree, which is equivalent to the following diagram being commutative:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"] & U \times F \arrow[dl, "p_1"] \\
U &
\end{tikzcd}$$

where φ is a homeomorphism.[13]

In fact, TM and T^*M are special kinds of fiber bundles: they are vector bundles. Vector bundles are fiber bundles where the fiber F has a vector space structure which is compatible with the above commutative diagram, in the sense that for any $q \in U$, φ defines a vector space isomorphism $\pi^{-1}(q) \mapsto M \times F$.

We can also consider maps between fiber bundles $\pi_1\colon E_1 \to M_1$ and $\pi_2\colon E_2 \to M_2$, which are just maps f, g that make the following diagram commute:

$$\begin{tikzcd}
E_1 \arrow[r, "f"] \arrow[d, "\pi_1"] & E_2 \arrow[d, "\pi_2"] \\
M_1 \arrow[r, "g"] & M_2
\end{tikzcd}$$

---

[12] The idea being that the sequence $F \hookrightarrow E \overset{\pi}{\to} M$ is exact as we want F to be the kernel of π.

[13] The set of all $\{U_i, \varphi_i\}$ is called the local trivialisation, which is the necessary data for π to be a fiber bundle.

To be a vector bundle morphism, we have to add the condition that the induced maps $\pi^{-1}(x) \to \pi^{-1}(g(x))$ are linear maps of vector spaces.
When $M_1 = M_2 = M$, we can consider the vector space $E_1 \oplus E_2$ and construct a corresponding vector bundle called the Whitney sum $\xi_1 \oplus \xi_2$ of ξ_1 and ξ_2, by setting $\pi\colon E_1 \oplus E_2 \to M$ by $\pi = (\pi_1, \pi_2)$, so that the fiber above each point is just the direct sum of the two fibers.
Given $\pi\colon TM \to M$, we can construct a vector field X by choosing, for each point in M, a vector in the fiber T_xM. For a general fiber bundle, this idea is that of a section:

Definition 3.7 Let $\xi = \pi\colon E \to M$ be a fiber bundle. A section of ξ is a map $s\colon M \to E$ such that $\pi(s(p)) = p$ for all $p \in M$. The set of all sections of ξ is denoted $\Gamma(E)$.

Then, we can define a (co)vector field as a section of the (co)tangent bundle. We would like to generalise cotangent vectors to functions that take more than one tangent vector and give out a real number, linear in each variable. This precisely corresponds to taking tensor products of cotangent vectors.

3.1.3 Tensors

Given two vector spaces U and V over a field K, we can form the vector space $U \otimes_K V$ as follows:
Consider the vector space W which has as basis elements of the form (u, v) with $u \in U$ and $v \in V$.
The vector space $U \otimes_K V$ (from now on $U \otimes V$) is then the quotient of W under the following identifications:
$(u_1 + u_2, v) = (u_1, v) + (u_2, v)$
$(u, v_1 + v_2) = (u, v_1) + (u, v_2)$
$(\lambda u, v) = (u, \lambda v) = \lambda(u, v)$ for $\lambda \in K$
We then write $u \otimes v$ for the image of (u, v) under the identification.
The most important feature of the tensor product is the following "universal property":
Given any bilinear map $f\colon U \times V \to Z$, there exists a unique linear map $p\colon U \otimes V \to Z$ that makes the following diagram commute:

$$\begin{tikzcd}
U \times V \arrow[r, "\otimes"] \arrow[dr, "f"'] & U \otimes V \arrow[d, "p"] \\
 & Z
\end{tikzcd}$$

This is because a bilinear map $f\colon U \times V \to Z$ can just be seen as a map $\tilde{f}\colon U \otimes V \to Z$ by $\tilde{f}(u \otimes v) = f(u, v)$, and the fact that f is bilinear implies precisely that the map $\tilde{f}$ is linear by the quotients we took to get $U \otimes V$ out of W. As all three quotients are necessary, we can see that we have constructed the most general space we could that has these properties, so this gives an idea of why this universal property holds.

We can therefore see the tensor product as being the "most general bilinear operation", as any bilinear map from $U \times V$ must factor through $U \otimes V$. Indeed, the above commutative diagram implies that $\mathrm{L}^2(U \times V, Z) \cong \mathrm{L}(U \otimes V, Z)$.

Definition 3.8 $T^m_n(V) = L^{m+n}(\underbrace{V^*, \dots, V^*}_{m}, \underbrace{V, \dots, V}_{n}; \mathbb{R}) \cong \underbrace{V \otimes \cdots \otimes V}_{m} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{n}$.[14]
Elements of T^m_n are called tensors of type (m, n).

Here we will be mainly interested in the case $V = T_x M$. We can then define the vector bundle of tensors of type (m, n) as $T^m_n(M) = \coprod_{x \in M} T^m_n(T_x M)$. Tensors fields are then smooth sections of $T^m_n(M)$: $\mathcal{T}^m_n = \Gamma(T^m_n)$. Vector fields are then elements of $\mathcal{T}^1_0$, and covector fields elements of $\mathcal{T}^0_1$. We can construct the tensor algebra as the space $T(M) = \bigoplus_{k=0}^{\infty} T^0_k(M)$, which is the algebra of tensors of type $(0, k)$ for all $k \in \mathbb{N}$. We can now take diverse quotients: the exterior algebra $\Lambda(M) = T(M)/I$, where I is the ideal generated by all elements of the form $x \otimes x$, gives us antisymmetric tensors.[15] We can similarly form the space of symmetric tensors $S(V)$ by taking I to be generated by elements of the form $x \otimes y - y \otimes x$.

The product in the exterior algebra $\Lambda(M)$ is given by the wedge product $x \wedge y = [x \otimes y]$ where $[x \otimes y]$ is the representative element of $x \otimes y$ under the quotient by I. [16]

$\Lambda(M)$ is a graded algebra $\Lambda^0(M) \oplus \Lambda^1(M) \oplus \cdots \oplus \Lambda^d(M)$[17] where d is the dimension of M. We then have the same structure on $\Omega(M) = \Gamma(\Lambda(M)) = \Omega^0(M) \oplus \Omega^1(M) \oplus \cdots \oplus \Omega^d(M)$, and elements of $\Omega^k(M)$ are called differential k-forms.

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[14] That is, the set of $m + n$-linear maps that take m vectors in V^*, n vectors in V, and give a real number. The equality on the right follows from the previous discussion of tensor products and their relation to multilinear maps.

[15] As quotienting out by the relation $x \otimes x = 0$ implies that $(x + y) \otimes (x + y) = 0$, which expanded gives $x \otimes x + y \otimes y + x \otimes y + y \otimes x = x \otimes y + y \otimes x = 0$.

[16] This is equivalent to the following definition: Let $\mathrm{Alt}(T(x_1, \dots, x_n)) = \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) T(x_{\sigma(1)}, \dots, x_{\sigma(n)})$, then $x \wedge y = \frac{(k+l)!}{k!l!} \mathrm{Alt}(x \otimes y)$ where x and y are k and l-differential forms, respectively. The coefficient $1/n!$ guarantees that $\mathrm{Alt}(\mathrm{Alt}(T)) = \mathrm{Alt}(T)$ as if T is already alternating, the sum is just the addition of $n!$ copies of T.
This definition shows the similarity differential forms share with the determinant. In fact, it can be shown that, at a point, all differential d-forms (where $d = \dim(M)$) are a scalar multiple of det. This explains the coefficient $(k + l)!/k!l!$ as in $\mathbb{R}^n$ this guarantees $dx_1 \wedge \cdots \wedge dx_n = \det$.

[17] This follows from the direct sum construction of $T(M)$. This just means that the wedge product of $x \in \Lambda^k(M)$ and $y \in \Lambda^l$ satisfies $x \wedge y \in \Lambda^{k+l}(M)$. The same works with the tensor algebra with the tensor product, or the symmetric algebra with the corresponding symmetric product $\mathrm{Sym}(T(x_1, \dots, x_n)) = \frac{1}{n!} \sum_{\sigma \in S_n} T(x_{\sigma(1)}, \dots, x_{\sigma(n)})$.

Hence differential k-forms are, at each point $x \in M$, an alternating k-linear map, which varies smoothly as x is moved. For example, differential one-forms on a 3 dimensional manifold would look locally like $\omega = f_1 dx_1 + f_2 dx_2 + f_3 dx_3$, where the f_i are smooth functions on M and the dx_i form the standard (local) coordinate basis of the cotangent space $T_x M$, which are dual to the vectors $\frac{\partial}{\partial x_i}$ induced from a chart.

Given any tensor field T on M, we can notice that T is linear over $C^\infty(M)$, the smooth functions on M. This is because, at each point, a tensor field is just a tensor and a function is just a number, so that multilinearity of the tensor implies that T is multilinear over the smooth functions. In fact, this is an essential property of tensor fields, as the following result shows:

Theorem 3.9 If $T\colon TM \times \cdots \times TM \times T^*M \times \cdots \times T^*M \to C^\infty(M)$ is multilinear over $C^\infty(M)$, then T is actually a tensor field.

Proof. Refer to [Spi99b]. ❖

This result allows us to see directly if a given map T defines a tensor field: we just have to check if it is linear over the smooth functions on M.[18]

3.1.4 Differential forms and the exterior derivative

If ω is a differential 0-form (that is, a smooth function f), we have the usual notion of derivative: $d\omega = \sum_i \frac{\partial f}{\partial x_i} dx_i$ which corresponds to the usual gradient of a function. It is possible to generalise this to differential k-forms in the following way: Let $\omega = f dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, then $d\omega = \sum_{j=1}^n \frac{\partial f}{\partial dx_j} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. Then the operator d is defined with this and the fact that $d(\omega + \psi) = d(\omega) + d(\psi)$ as every differential k-form must be the sum of such differential forms with only one component.

To make notation easier, we can use a multi-index notation: given a multi-index $I = (i_1, \ldots, i_k)$, write $f dx_I$ for $f dx_{i_1} \wedge \cdots \wedge dx_{i_k}$.

Theorem 3.10 For all differential forms ω, $d(d\omega) = 0$.

Proof. Again, suppose ω is of the form $\omega = f dx_I$. Then $d\omega = \sum_{j=1}^n \frac{\partial f}{\partial dx_j} dx_j \wedge dx_I$, and

$$d(d\omega) = \sum_{l=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_j \partial x_l} dx_j \wedge dx_I$$

---

[18] This is why $\mathcal{T}(M) = \Gamma(T(M))$ is sometimes referred to as a $C^\infty(M)$-module (a module is just an abelian group with a scalar multiplication on it by a ring (with some compatibility conditions), in the same way a vector space is an abelian group with scalar multiplication from elements in a field).

But the mixed second partial derivatives are equal, so we can switch round j and l, but as $dx_j \wedge dx_l = -dx_l \wedge dx_j$, we have that $d(d\omega) = -d(d\omega)$ so $d(d\omega) = 0$. As every differential k-form is a sum of such differential k-forms and as the exterior derivative is linear, we have that for all ω, $d(d\omega) = 0$. ❖

Theorem 3.11 If ω is a differential k-form, then $d(\omega \wedge \psi) = d\omega \wedge \psi + (-1)^k \omega \wedge d\psi$

Proof. Consider $\omega = f dx_I$ and $\psi = g dx_J$. We then have:

$$\begin{aligned} d\left(\omega \wedge \psi\right) &= d\left(fg dx_I \wedge dx_J\right) \\ &= d(fg) \wedge dx_I \wedge dx_J \\ &= ((df)g + f(dg)) \wedge dx_I \wedge dx_J \\ &= dfg \wedge dx_I \wedge dx_J + fdg \wedge dx_I \wedge dx_J \\ &= df \wedge dx_I \wedge g dx_J + (-1)^k f dx_I \wedge dg \wedge dx_J \\ &= d\omega \wedge \psi + (-1)^k \omega \wedge d\psi \end{aligned}$$

❖

This definition of exterior derivative, however, has the disadvantage of depending on a choice of coordinates dx_i, but is in fact uniquely determined by some of its properties we have just shown:

Proposition 3.12 Suppose $\delta\colon \Omega^k(M) \to \Omega^{k+1}(M)$ has the following properties:

$$\delta(\omega + \eta) = \delta(\omega) + \delta(\eta) \tag{1}$$

$$\delta(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\deg(\omega)} \omega \wedge d\eta \tag{2}$$

$$\delta(\delta(\omega)) = 0 \tag{3}$$

$$\delta(f) = df \quad \text{for } f \in C^\infty(M) \tag{4}$$

Then $\delta = d$

Proof. Consider $\omega = f dx_I$.
Then $\delta\omega = df \wedge dx_I + f \wedge \delta(dx_I)$.
For $d\omega = \delta\omega$, we need $\delta(dx_I) = 0$.
As the dx_i come from the coordinate functions, from (4) we have that $dx_i = \delta x_i$. As in [Spi99b], notice we can now proceed by induction: suppose $\delta(dx_{i_1} \wedge \cdots dx_{i_{k-1}}) = 0$.
Then $\delta(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \delta(dx_{i_1}) \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} - dx_{i_1} \delta(dx_{i_2} \wedge \cdots \wedge dx_{i_k})$. The first term is 0 because $\delta^2 = 0$ and the second term is 0 by the induction hypothesis. Therefore $\delta = d$.
In fact, this ensures uniqueness of the exterior derivative, as we started by defining an

unique d on differential forms that look like $dx_1 \wedge \cdots \wedge dx_k$, and seen that there is an unique δ that agrees with d. ❖

Remark 3.13 For later use, note that there is another definition of the exterior derivative, which is also independent of any coordinates:

$$\begin{aligned} d\omega(X_1, \dots, X_{n+1}) &= \sum_{k=1}^{n+1} (-1)^k X_k \omega(X_1, \dots, \hat{X}_k, \dots, X_{n+1}) \\ &+ \sum_{1 \leqslant i < j \leqslant n+1} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{n+1}) \end{aligned}$$

Where $[X, Y]$ is the Lie bracket of vector fields that we shall study later on.
This is shown to be equivalent to our usual definition in [Spi99b]. For example, this allows us to write $d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$.

Suppose we have a map between manifolds $f \colon M \to N$, we would like to relate the differential forms between the manifolds. We have an associated map between the tangent spaces at each point given by the pushforward of f, given by $f_* \colon T_x M \to T_{f(x)} N$. We can then consider the dual f^* of f_*, called the pullback, given by $f^*\omega(x_1, \dots, x_n) = \omega(f_*(x_1), \dots, f_*(x_n))$. Notice that the pullback of differential forms is much better behaved than the pushforward of vector fields, as the map $f \colon M \to N$ might not be injective for example, so we wouldn't know how to pushforward a tangent vector if it has two distinct preimages. Hence, whereas the pushforward only really applies to tangent vectors (and not vector fields), the pullback works for covector fields, and hence differential forms, so it becomes the more natural object to study.

Lemma 3.14 $f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)$

Proof.

$$\begin{aligned} & f^*(\omega \wedge \eta)(v_1, \dots, v_k, v_{k+1}, \dots, v_{k+l}) \\ &= (\omega \wedge \eta)\left(f_*(v_1), \dots, f_*(v_k), f_*(v_{k+1}), \dots, f_*(v_{k+l})\right) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sign}(\sigma) \omega\left(f_*(v_1), \dots, f_*(v_k)\right) \eta\left(f_*(v_{k+1}), \dots, f_*(v_{k+l})\right) \\ &= (f^*(\omega) \wedge f^*(\eta))(v_1, \dots, v_k, v_{k+1}, \dots, v_{k+l}) \end{aligned}$$

❖

Remark 3.15 $f^*(dx_i)(X) = dx_i(f_*(X)) = (f_*(X))x_i = X(x_i \circ f) = d(x_i \circ f)(X) = df_i(X)$ where f_i is the i-th coordinate of f corresponding to the x_i in the chart, so $f^*(dx_i) = df_i$.

Theorem 3.16 Pullbacks commute with exterior derivatives. That is, for $f\colon M \to N$ and a differential form ω, $f^*(d\omega) = d(f^*(\omega))$.[19]

Proof. Consider the case $\omega = g \in \Omega^0(M)$. By a similar reasoning than in Remark 3.15, we have that $f^*(dg) = d(g \circ f) = d(f^*g)$ which is precisely what we wanted. Now in general, consider $\omega = g \wedge dx_I$. Then $d\omega = dg \wedge dx_I$ by (2) of Proposition 3.12, as $d(dx_I) = 0$. Therefore we have that:

$$\begin{aligned} f^*(d\omega) &= f^*(dg \wedge dx_I) \\ &= f^*(dg) \wedge f^*(dx_I) \quad \text{By Lemma 3.14} \end{aligned}$$

But $d(f^*(dx_I)) = d(df_i) = 0$ by Remark 3.15 and the fact that $d \circ d = 0$, so $d(f^*(g) \wedge f^*(dx_I)) = d(f^*(g)) \wedge f^*(dx_I)$, hence $f^*(d\omega) = d(f^*(g) \wedge f^*(dx_I)) = d(f^*(\omega))$. ❖

3.1.5 de Rham cohomology

Consider the "de Rham complex"

$$0 \to \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} \Omega^n(M) \to 0$$

Where the d_k are just the usual exterior derivative of k-forms. We know that $d^2 = 0$, so that in each case $\operatorname{im}(d_{k-1}) \subset \ker(d_k)$. Both of these are subgroups of the (abelian) group $\Omega^k(M)$, so we can consider the quotient $H^k(M) = \operatorname{im}(d)/\ker(d)$, called the k-th cohomology group of M. Differential forms ω with $d\omega = 0$ are called closed and differential forms ω with a differential form ψ such that $d\psi = \omega$ are called exact. The idea of de Rham cohomology is then to measure the extent to which closed differential forms are exact. This stems from the idea of considering vector fields in $\mathbb{R}^3$, for example we know that on a simply-connected domain, every irrotational vector field $V = (v_1, v_2, v_3)$ is the gradient of some scalar potential f. This translates to the language of differential forms as follows: write $\omega = v_1 dx_1 + v_2 dx_2 + v_3 dx_3$. V is irrotational, so that $\nabla \times V = 0$, but $d\omega$ corresponds to $\nabla \times V$ because $d\omega = \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right) dx_2 \wedge dx_3 + \left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right) dx_3 \wedge dx_1 + \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right) dx_1 \wedge dx_2$, so $d\omega = 0$. As V is the gradient of f, we know that we can write $\omega = d\psi$ with $\psi = f$ as $d\psi = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3$. Then $H^1(M) = 0$ as all closed 1-forms are exact. The fact that this always happens on simply connected domains in $\mathbb{R}^n$ is known as the Poincaré

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[19] In fact, by a paper of Palais [Pal59], the exterior derivative is the only map $\Omega^k(M) \to \Omega^{k+1}(M)$ with this property, up to scalar multiplication.

Lemma.

However, on more complicated domains this doesn't always happen: consider the polar coordinate system (r, θ) on $M = \mathbb{R}^2 \setminus 0$. We can write locally $\theta = \arctan(y/x)$, and then $d\theta$ is a well-defined 1-form on M, $d\theta = \frac{-ydx}{x^2+y^2} + \frac{xdy}{x^2+y^2}$, but θ is not a valid 0-form as it is not continuous (or multivalued). Then $d(d\theta) = 0$ but $d\theta$ is not exact (despite the rather confusing notation $d\theta$), so this gives us a nontrivial element $[d\theta] \in H^1(M)$, and in fact it "generates" it, so that $H^1(M) \cong \mathbb{R}$.

One can also consider compactly supported cohomology, which arises in the exact same way but considering only compactly supported differential forms ω, so that $\operatorname{supp}(\omega)$ is contained in some compact set, and we then write $H^k_c(M)$ for the k-th compactly supported cohomology group.

3.1.6 Lie groups

Many manifolds happen to also have a group structure, for instance, many matrix groups are manifolds: $\mathrm{GL}(n, \mathbb{R})$ can be seen to be a manifold by observing that $\det \colon \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}$ is a continuous map (it's a polynomial in the matrix entries), so the preimage of any open subset of $\mathbb{R}$ is open. By considering $\det^{-1}(\mathbb{R} \setminus 0)$ as an open subset of $\mathbb{R}^{n^2}$, we see that $\mathrm{GL}(n, \mathbb{R})$ is an n^2-dimensional manifold, as every nonempty open subset of $\mathbb{R}^m$ can be considered as an m dimensional manifold with charts being the identity.

This correspondance between manifolds and groups leads to the following definition:

Definition 3.17 A Lie group is a smooth manifold G equipped with a group structure,[20] in which the operation of group multiplication and inversion are smooth maps from G to itself.

As was seen above, $\mathrm{GL}(n, \mathbb{R})$ is a Lie group.[21] $\mathrm{O}(n, \mathbb{R})$ is also a Lie group, as can be shown by considering the map $A \mapsto A^t A$.

Given any Lie group G, we can consider vector fields on G as with usual manifolds. But we now have a way to relate different points of our manifolds, by the group operation: we have a left translation L_g corresponding to left multiplication by g, and a right translation R_g corresponding to right multiplication by g. In particular, given points g and h in G, we can move a tangent vector at g to one at h by saying $X_h = L_*(hg^{-1})(X_g)$, where L_*

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[20] We will write the group operation multiplicatively, and the identity element will be denoted e.

[21] In fact, this requires a little bit more work, as we need to show that the group structure is compatible, in that multiplication and inversion are smooth maps. This can in fact be seen that for matrix multiplication, all elements are given by polynomial operations, so is smooth, and the inversion can also be seen to be smooth by considering the usual formula for an inverse by the method of cofactors (Cramer's rule).

is the pushforward of the left translation. This motivates the definition of left-invariant vector fields:

Definition 3.18 A left invariant vector field X on a Lie group G is a vector field that satisfies $X_g = (L_g)_* X_e$.

This means that a left invariant vector field is a vector field that is compatible with the group structure, as we get the same vector field whether we consider X_g or the induced image of X_e under the map $L_g\colon e \to g$. This allows us to form a pairing between T_eG and the set of all left-invariant vector fields on G, as given a tangent vector $X \in T_eM$ we can develop it into a left invariant vector field precisely by setting $X_g = (L_g)_* X$.
Given two vector fields X and Y, in addition to be able to add them together, we have a natural product called the Lie bracket $[X, Y]$ given by $[X, Y](f) = X(Y(f)) - Y(X(f))$ by considering vector fields as acting on smooth functions as derivations.[22] This is indeed another vector field as, at each point, we have that

$$\begin{aligned}
[X,Y](fg) &= X(Y(fg)) - Y(X(fg)) \\
&= X(fY(g) + gY(f)) - Y(fX(g) + gX(f)) \\
&= fX(Y(g)) - fY(X(g)) + gX(Y(f)) + gY(X(f)) \quad \text{as } X(fY(g)) = fX(Y(g)) \\
&= f[X,Y](g) + g[X,Y](f) \qquad\qquad \text{because, at a point, } f \text{ is just a number}
\end{aligned}$$

This calculation shows us why the natural product operation to consider is the Lie bracket rather than, say, $(X \times Y)(f) = X(Y(f))$ as $(X \times Y)(fg)$ is not a derivation.
This product of vector fields then transfers to the tangent space T_eG, by letting $[X, Y]$ be the Lie bracket of the corresponding left-invariant vector fields. Then T_eG becomes an algebra (a vector space with a (bilinear) product), called the Lie algebra of G, denoted $\mathfrak{g}$.
As it is possible to consider $\mathfrak{g}$ as T_eG, we can also use the information in $\mathfrak{g}$ to get a Lie group. Given a tangent vector, we know we can form a left-invariant vector field X that agrees with the tangent vector at the identity. We can then consider so called integral curves of X, which are the curves γ satisfying $\gamma_*(\frac{\partial}{\partial t}) = X$.

Theorem 3.19 There exists an unique map $\exp\colon \mathfrak{g} \to G$ such that $\exp(tX)$ for $t \in \mathbb{R}$ is an integral curve for X with $\exp(0) = e$.

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[22] This Lie bracket is sometimes called the Lie derivative $\mathcal{L}_X Y = [X, Y]$ as it gives a measure of the rate of change of Y along X. This is intimately related with the study of connections, as we will soon see. But the Lie derivative can be define for other objects than vector fields: we can define if for functions ($\mathcal{L}_X f = X(f)$), and for differential forms, using the contraction operator $(i_X\omega)(X_1, \dots, X_n) = \omega(X, X_1, \dots, X_n)$, by $\mathcal{L}_X\omega = i_X d\omega + d(i_X\omega)$ which corresponds to $\mathcal{L}_X f = X(f) = df(X) = i_x df$, but taking into account the fact that X changes.

Proof. Omitted, refer to [Spi99b]. ❖

Thus the exponential map exp can be seen as a natural way to retrieve G from $\mathfrak{g}$ as we try to move along G by a curve uniquely determined by X. The exponential map then satisfies many familiar properties: $\exp(-aX) = (\exp(aX))^{-1}$ and $\exp((a_1 + a_2)X) = \exp(a_1X)\exp(a_2X)$. In fact, if we are considering matrix groups, the exponential map is just the usual exponential of a matrix given by $\exp(X) = \sum_{k=0}^{\infty} \frac{X^k}{k!}$, as it indeed defines an integral curve $\gamma(t) = \exp(tX)$ as $\gamma_*(\frac{\partial}{\partial t})(a) = \exp(aX)$ and $X(\gamma(a)) = X(\exp(aX)) = \exp(aX)$ by seeing X as a derivation.

If $G = \mathrm{GL}(n, \mathbb{R})$, we write $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R})$, and it can be shown that we then have that $[A, B] = AB - BA$ as a product of matrices. We also write $\mathfrak{so}(n, \mathbb{R})$ for the Lie algebra of $\mathrm{SO}(n, \mathbb{R})$, which is the same as the Lie algebra of $\mathrm{O}(n, \mathbb{R})$ as $\mathrm{O}(n, \mathbb{R})$ has two disconnected components (matrices with determinant 1 and -1), the one containing the identity being $\mathrm{SO}(n, \mathbb{R})$.

Proposition 3.20 $\mathfrak{so}(n, \mathbb{R})$ is the vector space of real $n \times n$ skew-symmetric matrices.

Proof. Suppose X is skew symmetric so $X^T = -X$. Then $\exp(aX^T) = \exp(-aX)$ so $\exp(aX)\exp(aX)^T = 1$ so $\exp(aX) \in \mathrm{O(n, \mathbb{R})}$ for all $a \in \mathbb{R}$, so that the skew-symmetric matrices are a subset of the matrices in $\mathfrak{so}(n, \mathbb{R})$. Conversely, suppose $X \in \mathfrak{so}(n, \mathbb{R})$, then $\exp(aX)^T \exp(aX) = I_n$ by definition. We can then write $\exp(X) = I_n + X + \frac{1}{2}X^2 + \cdots$, so we get $(I_n + aX^T + \frac{1}{2}a(X^2)^T + \cdots)(I_n + aX + \frac{1}{2}aX^2 + \cdots) = I_n$. But expanding out the product, we then get that $I_n = I_n + a(X^T + X) + \frac{1}{2}a^2((X^2)^T + 2X^TX + X^2) + \cdots$. As this is valid for all a, we must have that all terms except I_n on the right are 0, in particular, $X^t + X = 0$. Hence all matrices in $\mathfrak{so}(n, \mathbb{R})$ are skew-symmetric, which proves the implication the other way. ❖

This allows an easy computation of the dimension of $\mathrm{SO}(n, \mathbb{R})$: it is the same as that of the skew-symmetric matrices of dimension n. But a skew symmetric matrix is determined uniquely by the elements above the diagonal, and any choice of entries is valid. Hence we have $\frac{n(n-1)}{2}$ choices so $\dim(\mathrm{SO}(n, \mathbb{R})) = \frac{n(n-1)}{2}$.

Given any element $g \in G$, we can consider the smooth function $\Psi_g(h) = g^{-1}hg$ as an isomorphism $G \to G$, also called an automorphism of G. We then write $\Psi \colon G \to \mathrm{Aut}(G)$. As $g^{-1}eg = e$, this induces a map $\mathrm{Ad}_g \colon \mathfrak{g} \to \mathfrak{g}$ by $\mathrm{Ad}_g = (\Psi_g)_*$, hence $\mathrm{Ad} \colon G \to \mathrm{GL}(\mathfrak{g})$. Ad is called the adjoint representation of $\mathfrak{g}$.[23] Now given the adjoint representation $\mathrm{Ad} \colon G \to \mathrm{GL}(\mathfrak{g})$, we can again make that correspond to a map from $\mathfrak{g}$ by $\mathrm{ad} = (\mathrm{Ad})_*$

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[23] In general, a representation ρ is a homomorphism $\rho \colon G \to \mathrm{GL}(V)$. Hence Ad is indeed a representation.

so ad: $\mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ where $\mathrm{End}(\mathfrak{g})$ is just the set of all homomorphisms from $\mathfrak{g}$ to itself (called endomorphisms).

Proposition 3.21 ad is given by $\mathrm{ad}_x(y) = [x, y]$.

Proof. Omitted, see for example [Bum04]. ❖

3.1.7 The metric tensor

Another useful object we had for surfaces was the first fundamental form. We want something that generalises this to higher dimensions, we want an inner product. We know that inner products on a vector space are in bijection with symmetric positive-definite matrices, this helps us to provide a definition of a metric tensor on a manifold.

Definition 3.22 A metric tensor g on a manifold M is a tensor field of type $(0, 2)$ such that, for each point $x \in M$, g is symmetric and positive-definite, and g varies smoothly as x is moved.[24]

In local coordinates, $g = \sum_{i,j} g_{ij} dx_i \otimes dx_j$. We will also write g^{ij} for the matrix inverse of g_{ij}, so that $\sum_{i,j} g^{ij} g_{ij} = \delta_i^j$.

Theorem 3.23 Every manifold admits a Riemannian metric.

Proof. Omitted, see for example [Mor01]. It is essential that M be second countable, as partitions of unity are fundamental to this construction. ❖

3.2 Connections

Let ξ with $\pi : E \to M$ be a fiber bundle. We know that E looks locally like $U \times F$ for small enough U, but in general $E \neq M \times F$.

In the case $E = M \times F$, for each point $e = (m, f)$ we can write $T_e E = T_p M \oplus T_f F$, so any tangent vector X in $T_e E$ can be decomposed as a sum $X = X_h + X_v$ where X_h is the "horizontal" component of X corresponding to $T_m M$ and X_v is the "vertical" component corresponding to $T_f F$.

We would like to mimic such a situation for a general fiber bundle, so we define the vertical bundle $V = \ker(\pi_*)$.[25] At this point there doesn't seem to be a natural way of distinguishing vectors as "horizontal", as all we have are the local trivialisations of E as $U \times F$, but there is no right choice of trivialisation. So we define a connection to be precisely those horizontal vectors:

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[24] Equivalently, a metric g is a smooth map $g\colon TM \times TM \to \mathbb{R}$ such that at each point $g\colon T_x M \times T_x M \to \mathbb{R}$ is symmetric and positive definite.

[25] Therefore at a point $e = (m, f)$, V_e is just the set of tangent vectors to the fiber over m of π.

Definition 3.24 An Ehresmann connection $\mathcal{H}$ on a fiber bundle ξ is, for each point $e \in E$, a choice of subspace H_e of T_eE such that $T_eE = V_e \oplus H_e$ and that this choice $e \mapsto H_e$ is smooth.

Proposition 3.25 Every fiber bundle admits an Ehresmann connection.

Proof. Choose a Riemannian metric on E and define H_e to be the horizontal complement of V_e. This is a smooth choice of horizontal vectors as the metric forms a C^∞ tensor field. ❖

Now that we have a notion of horizontal vectors we can lift a curve γ in M up to a curve in E such that the tangent vector $\gamma_*(\frac{\partial}{\partial t})$ is always horizontal, giving a preferred way of "lifting" γ.

More precisely, let $\gamma\colon [a,b] \to M$ be a smooth curve, a curve $c\colon [a,b] \to E$ is called a lift of γ iff $\pi(c(t)) = \gamma(t)$ for all $t \in [a,b]$. If $c_*(\frac{\partial}{\partial t}) \in H_{c(t)}$ for all t, c is a horizontal lift of γ.

Theorem 3.26 Suppose $\gamma : [a,b] \to M$ is a piecewise C^∞ curve. Then for any point e_a in the fiber above $\gamma(a)$ there exists an unique horizontal lift c of γ such that $c(a) = e_a$

Proof. Omitted. See, for example, [Mor01]. ❖

This parallel transport is the reason for the word "connection": if we consider a connection on a manifold M (which just means that $E = TM$), given a curve $\gamma\colon [a,b] \to M$, we can lift it (horizontally) to a curve $c\colon [a,b] \to TM$, which allows us to slide the tangent space T_aM to that of T_bM by $\tau_{\gamma(t)}(X) = c(t)$ for $t \in [a,b]$ and $X \in T_aM$, hence "connecting" the tangent spaces.

We might have some additional structure on the fiber bundle which we would like to use: for example, ξ might be a vector bundle, as is the case when $E = TM$. This is the idea of a Koszul connection.

Definition 3.27 A Koszul connection on a vector bundle $\pi\colon E \to M$ is a map $\nabla\colon \Gamma(E) \to \Gamma(E \otimes T^*M)$ which is $\mathbb{R}$-linear and satisfies the Leibniz rule $\nabla(fY) = f\nabla Y + Y \otimes df$ for $Y \in \Gamma(E)$ and $f \in C^\infty(M)$.

Proposition 3.28 Every vector bundle admits a Koszul connection.

Proof. Omitted. See, for instance, [Mor01]. ❖

Proposition 3.29 Given a vector field $X \in TM$, a Koszul connection ∇ induces a covariant derivative $\nabla_X\colon \Gamma(E) \to \Gamma(E)$ by $\nabla_X(Y) = (\nabla Y)(X)$.

Proposition 3.30 The covariant derivative ∇_X satisfies the following properties:[26]

$$\begin{aligned} \nabla_{X_1+X_2}(Y) &= \nabla_{X_1}(Y) + \nabla_{X_2}(Y) && (1)\\ \nabla_X(Y_1+Y_2) &= \nabla_X(Y_1) + \nabla_X(Y_2) && (2)\\ \nabla_{fX}(Y) &= f\nabla_X Y && (3)\\ \nabla_X(fY) &= f\nabla_X Y + X(f)Y && (4) \end{aligned}$$

Proof. Property (2) follows from $\mathbb{R}$-linearity of ∇, and properties (1) and (3) follow from the fact that ∇_X is obtained by contraction of tensors (letting a 1-form act on a vector field) which is C^∞-linear.
(4) follows as $\nabla_X(fY) = \nabla(fY)(X) = (f\nabla Y + Y \otimes df)(X) = f\nabla(Y)(X) + (Y \otimes df)(X) = f\nabla_X(Y) + Y df(X) = f\nabla_X(Y) + X(f)Y$ by considering X as a derivation. ❖

These properties show that ∇ is $C^\infty(M)$-linear in X but only $\mathbb{R}$-linear in Y, so it does not define a tensor field, which would have to satisfy $\tilde{\nabla}_X(fY) = f\tilde{\nabla}_X(Y)$. However, given $Y \in \Gamma(E)$, the function $\nabla \colon X \mapsto \nabla_X(Y)$ does define a tensor field.

Definition 3.31 A linear Ehresmann connection $\mathcal{H}$ on a vector bundle $\pi \colon E \to M$ is an Ehresmann connection whose parallel translations (via horizontal lifts of γ) $\tau_{\gamma(t)}$ are $\mathbb{R}$-linear. That is, for $\lambda \in \mathbb{R}$, X, X_1, X_2 in $T_{\gamma(a)}E$, $\tau_{\gamma(t)}$ satisfies:

$$\begin{aligned} \tau_{\gamma(t)}(X_1 + X_2) &= \tau_{\gamma(t)}(X_1) + \tau_{\gamma(t)}(X_2)\\ \tau_{\gamma(t)}(\lambda X) &= \lambda \tau_{\gamma(t)}(X) \end{aligned}$$

Proposition 3.32 A linear Ehresmann connection $\mathcal{H}$ on a vector bundle $\pi \colon E \to M$ induces a Koszul connection.

Proof. An Ehresmann connection is precisely what is needed to be able to define a derivative along a curve "in the usual way", as it allows identification of tangent spaces by parallel translation τ. That is, for $X_p \in T_pM$, $Y \in \Gamma(E)$ and $\gamma : [a,b] \to M$ with $\gamma(a) = p$, $\gamma_*\left(\frac{\partial}{\partial t}\right)(a) = X$, we can define a covariant derivative ∇_{X_p} by

$$\nabla_{X_p}(Y) = \lim_{t \to 0} \frac{1}{t}\left((\tau_{\gamma(a+t)})^{-1}\left(Y\left(\gamma(a+t)\right)\right) - Y\left(\gamma(a)\right)\right)$$

Where $Y\left(\gamma(a+t)\right)$ means the the point $y \in Y : \pi(y) = \gamma(a+t)$. This is completely analogous to the usual definition of derivative as we can consider Y to be the graph of a function

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[26] These are precisely the properties of a (Koszul) connection as defined in [Spi99a]. It can be shown that any operator satisfying these properties is a Koszul connection in the sense defined above.

over M, and if we take γ to be the identity we get $\lim_{t\to 0} \frac{1}{t}\left(\left(\tau_{a+t}\right)^{-1}\left(Y\left(a+t\right)\right)-Y\left(a\right)\right)$ which is the usual definition modulo the needed parallel transport to identify points in the fiber over $a+t$ to points in the fiber over a.

We now only have to check properties (1)-(4) of Proposition 3.30 to check that this defines a Koszul connection on E:

(1) is satisfied as pushforwards are linear maps.

(3) is satisfied as the definition of $\nabla_X(Y)$ is given pointwise with respect to X, so fX is just $f(p)X_p$ at each point, so we can again use that the pushforward is a linear map.

(2) follows from linearity of $\mathcal{H}$.

(4) is a consequence of the product rule for derivatives, using that $\nabla_X(\lambda Y) = \lambda\nabla_X Y$ by linearity of $\mathcal{H}$. ❖

In coordinates, we can express the Koszul connection ∇ by its action on a basis: this defines the connection coefficients Γ_{ij}^k by $\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j} = \sum_{i,j}\Gamma_{ij}^k\frac{\partial}{\partial x_k}$.

As [Lee97] notes, every submanifold of $\mathbb{R}^n$ possesses an induced connection given by $i_*\left(\nabla_X Y\right) = \left(\overline{\nabla}_X Y\right)_H$ where i is the inclusion $i\colon M\to\mathbb{R}^n$, H denotes the orthogonal projection of $T_p\mathbb{R}^n$ onto T_pM and $\overline{\nabla}$ is the standard connection on $\mathbb{R}^n$, given by $\nabla_X\left(\sum_i a_i\frac{\partial}{\partial x_i}\right) = \sum_i (Xa_i)\frac{\partial}{\partial x_i}$, which is just the ordinary directional derivative; note that the X, Y appearing on the right hand side are just X and Y on M that we have "extended" in a neighbourhood of M.

Given a vector bundle $\pi\colon E\to M$ we can consider E-valued differential forms (called vector valued differential forms) $\Omega^k(M;E) = \Gamma(\Lambda^k T^*M\otimes E)$. For a fixed vector space V, a V-valued differential form is an E-valued differential form for $E = M\times V$. For a V-valued differential form ω, we can simply define an exterior derivative $d\omega$ by the action of d on a basis: say $\omega = \sum_i \alpha_i\omega_i$, then $d\omega = \sum\alpha_i d\omega_i$. But for a general vector valued differential form, we cannot do the same thing as the vector bundle may exhibit nontrivial global features, because the local trivialisations are local by definition. The following definition corrects this defect.

Definition 3.33 A covariant exterior derivative on a vector bundle $\pi\colon E\to M$ is a set of maps $d_k^E\colon\Omega^k(M;E)\to\Omega^{k+1}(M;E)$ that satify the Leibniz rule $d^E(\omega\wedge\eta) = d^E(\omega)\wedge\eta + (-1)^k\omega\wedge d\eta$, where $\omega\in\Omega^k(M;E)$ and $\eta\in\Omega^l(M)$.[27]

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[27] The wedge product is here defined as $\wedge\colon\Omega^p(M;E_1)\times\Omega^q(M;E_2)\to\Omega^{p+q}(M;E_1\otimes E_2)$ by

$$\omega\wedge\eta(x_1,\dots,x_{p+q}) = \frac{1}{p!q!}\sum_{\sigma\in S_{p+q}}\operatorname{sign}(\sigma)\omega\left(x_{\sigma(1)},\dots,x_{\sigma(p)}\right)\otimes\eta\left(x_{\sigma(p+1)},\dots,x_{\sigma(p+q)}\right)$$

This is the same formula than for usual $\mathbb{R}$-valued differential forms, except with multiplication replaced by the tensor product. Hence in this definition, the wedge product gives another E-valued

As $\Gamma(E \otimes \mathbb{R}) = \Gamma(E)$ and because $\Omega^k(M; E) = \Gamma(E \otimes \Lambda^k T^*M)$, we can see that a connection is also a map $\nabla\colon \Omega^0(M; E) \to \Omega^1(M; E)$. In fact this connection is the most natural way to generalise the exterior derivative (which becomes the special case when E is the trivial bundle $M \times \mathbb{R}$):

Theorem 3.34 Given a Koszul connection on a vector bundle $\pi\colon E \to M$, there is an unique covariant exterior derivative d_∇ such that $d_\nabla = \nabla$ for 0-forms and $d_\nabla(\omega \wedge \eta) = d(\omega) \wedge \eta + (-1)^k \omega \wedge (d_\nabla(\eta))$ for $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M; E)$.

Proof. Given $\omega \in \Omega^k(M)$ and $\sigma \in \Omega^0(M; E)$, define d_∇ as $d_\nabla(\omega \otimes \sigma) = d\omega \wedge \sigma + (-1)^k \omega \wedge \nabla\sigma$. As $\sigma \in \Omega^0(M; E)$ with $k = 0$ we have that $d\omega \wedge \sigma = d\omega \otimes \sigma$ so that $\nabla(\omega \wedge \sigma) = d\omega \otimes \sigma + (-1)^k \omega \wedge \nabla\sigma$, and hence $d_\nabla = \nabla$ for $k = 0$. For the other property, with $\eta \otimes \sigma \in \Omega^l(M; E)$, we have that:

$$\begin{aligned} d_\nabla(\omega \wedge (\eta \otimes \sigma)) &= d_\nabla((\omega \wedge \eta) \otimes \sigma) = d(\omega \wedge \eta) \otimes \sigma + (-1)^{k+l}(\omega \wedge \eta) \wedge \nabla\sigma \\ &= (d\omega \wedge \eta) \otimes \sigma + ((-1)^k \omega \wedge d\eta) \otimes \sigma + (-1)^{k+l}(\omega \wedge \eta) \wedge \nabla\sigma \\ &= d\omega \wedge (\eta \otimes \sigma) + (-1)^k \omega \wedge d_\nabla(\eta \otimes \sigma) \end{aligned}$$

❖

Hence the covariant exterior derivative is the natural generalisation of the exterior derivative for vector valued differential forms. And while the Leibniz rule is still present, we might wonder what happened to the fact $d^2 = 0$. This is no longer true with vector valued differential forms as the following example shows:

Given $X \in \Gamma(TS^2)$, we have a map $f\colon S^2 \to \mathbb{R}^2$ given by $f\colon p \mapsto X_p$, and satisfying $\langle f(p), N_p \rangle = 0$, where N_p is the normal vector to S^2 at p, which can be considered as p itself in $\mathbb{R}^3$. This allows us to define a connection $\nabla_X Y = dY(X) + \langle X, Y \rangle N$ at each point p, which one can easily check to be a connection. Then, with $\omega \in \Omega^0(M, TM)$, $d_\nabla \circ \nabla(\omega)(X) = d_\nabla\left(d\omega(X) + N \cdot g(\omega, X)\right)$ so that $d_\nabla \circ \nabla = dN \wedge g \neq 0$, which we can notice is related to the Gaussian curvature as dN is analogous to $d\nu$.

In fact, the nonvanishing of d_∇^2 is a consequence of the "curvature" of the fiber bundle $\pi\colon E \to M$ which isn't necessarily trivial, as the above example hints.[28] This curvature can be measured by the curvature of the Koszul connection.

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differential form. If ω and η are $\mathfrak{g}$-valued differential forms, we can define $[\omega, \eta]$ as the composite $\Omega^k(M, \mathfrak{g}) \times \Omega^l(M, \mathfrak{g}) \xrightarrow{\wedge} \Omega^{k+l}(M, \mathfrak{g} \otimes \mathfrak{g}) \xrightarrow{[\cdot,\cdot]} \Omega^{k+l}(M, \mathfrak{g})$ where the bilinear $[\cdot,\cdot]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ can be considered as the linear map $[\cdot,\cdot]\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ by $[X \otimes Y] = [X, Y]$.

[28] In fact, TS^2 is necessarily nontrivial, for example as a consequence of the Hairy Ball Theorem in topology which states that there can be no continuous nowhere-zero tangent vector field on S^2

3.3 The curvature tensor

Remember that with usual differential forms, the equality $d^2 = 0$ was a consequence of the equality of the mixed partial derivatives. In this light, seeing as the connection ∇ makes precise the idea of directional derivative with vector-valued differential forms, we can consider $S(X,Y)(Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z$. However we might also have $[X,Y] \neq 0$, which is something we don't want S to measure. Hence it is more appropriate to consider $R(X,Y)(Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$.

Theorem 3.35 $R(X,Y)(Z)$ is a tensor field.

Proof. It suffices to show that R is $C^\infty(M)$-linear in all three variables.

$$\begin{aligned}
R(fX,Y)(Z) &= \nabla_{fX}\nabla_Y Z - \nabla_Y \nabla_{fX} Z - \nabla_{[fX,Y]}(Z)\\
&= f\nabla_X \nabla_Y Z - \nabla_Y(f\nabla_X Z) - \nabla_{[fX,Y]}(Z)\\
&= f\nabla_X\nabla_Y Z - f\nabla_Y\nabla_X(Z) - Y(f)\nabla_X(Z) - \nabla_{f[X,Y]-Y(f)X}(Z)\\
&= f\nabla_X\nabla_Y Z - f\nabla_Y\nabla_X(Z) - f\nabla_{[X,Y]}(Z)\\
&= fR(X,Y)(Z)
\end{aligned}$$

and similarly for Y. For Z:

$$R(X,Y)(fZ) = \nabla_X\nabla_Y fZ - \nabla_Y\nabla_X fZ - \nabla_{[X,Y]}(fZ)$$

$$\begin{aligned}
\nabla_X\nabla_Y fZ &= \nabla_X(f\nabla_Y(Z)) + \nabla_X(Y(f)(Z))\\
&= f\nabla_X\nabla_Y Z + X(f)\nabla_Y Z + Y(f)\nabla_X Z + X(Y(f))Z
\end{aligned}$$

Similarly

$$\nabla_Y\nabla_X Z = f\nabla_Y\nabla_X Z + X(f)\nabla_Y Z + Y(f)\nabla_X Z + Y(X(f))Z$$

$$\begin{aligned}
\nabla_{[X,Y]} fZ &= f\nabla_{[X,Y]}(Z) + [X,Y](f)(Z)\\
&= f\nabla_{[X,Y]}(Z) + (XY(f) - YX(f))Z
\end{aligned}$$

Hence $R(X,Y)(fZ) = fR(X,Y)(Z)$, so R defines a tensor field on M. ❖

We also see how the term $\nabla_{[X,Y]}Z$ is important, as $S(X,Y)$ does not define a tensor field.

We are now ready to see how the non-vanishing of the composition $d_\nabla \circ d_\nabla$ is related to the curvature of our space. Remember that for (usual) differential 1-forms we had that $d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$. We have an analogous formula for a Koszul connection, $(d_\nabla\omega)(X, Y) = \nabla_X\omega(Y) - \nabla_Y\omega(X) - \omega([X, Y])$, which follows from Theorem 3.34 and Remark 3.13.

Theorem 3.36 $(d_\nabla)^2(\omega) = R \wedge \omega$

Proof. First suppose $\omega \in \Omega^0(M; E)$. Then we have that:

$$\begin{aligned}(d_\nabla)^2(\omega)(X, Y) &= \nabla_X(d_\nabla\omega(Y)) - \nabla_Y(d_\nabla\omega(Y)) - d_\nabla\omega([X, Y]) \\ &= \nabla_X\nabla_Y\omega - \nabla_Y\nabla_X\omega - \nabla_{[X,Y]}\omega = R(X, Y)\omega = R(X, Y) \wedge \omega\end{aligned}$$

Then in general, we can use the properties of d_∇ to show that this indeed holds for all ω (refer to [Dar94] or [MT97] for details). ❖

In fact, many authors use this theorem to define the curvature of a connection; this is the case in [MT97] for example.

3.4 Torsion of a connection

In a similar way to our definition of the curvature, if $E = TM$, we can consider the torsion of a connection given by $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$.[29]

The fact that $E = TM$ allows us to give a more geometric explanation of curvature and torsion: the curvature measured how the tangent space "rolled" when being parallel transported, as the tangent vector is turned around in the presence of curvature; the torsion measures rather how the tangent space "twists" around the curve along which it is transported, which can be pictured by looking at two coplanar line segments (vectors) in $\mathbb{R}^3$: put them at the same end, and parallel transport one along the other, and then start again by parallel transporting the other one. If there was no torsion, it should be the case that by putting the two cases together, we get a parallelogram, but with torsion, the two ends might fail to meet up as one line segment could have rotated around the other during the transport. This is exactly the non-commutativity measured by $\nabla_X Y - \nabla_Y X$,

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[29] This links back to our considerations about the Lie derivative $\mathcal{L}_X Y = [X, Y]$: the torsion tensor precisely measures the difference between the Lie derivative and the analogous concept with connections. As noted in [Boo86], the operators ∇ and $\mathcal{L}$ share many properties, but while we can ask for $\nabla_{X_p} Y$ for X at a point p, the Lie derivative $\mathcal{L}_X Y$ doesn't "live at points" as it asks for vector fields. Indeed, $[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$ so the Lie derivative is not tensorial; this shows that these are two different paradigms for differentiation. However $\nabla_X Y - \nabla_Y X$ is also non-tensorial, as we know from the properties of ∇.

the correction term $[X, Y]$ is really as before, and serves to make sure $T(X, Y)$ is a tensor field:

Proposition 3.37 $T(X, Y)$ is a tensor field.[30]

Proof.

$$\begin{aligned} T(fX, Y) &= \nabla_{fX}Y - \nabla_Y(fX) - [fX, Y] \\ &= f\nabla_X Y - f\nabla_Y X - Y(f)X - fX(Y) + Y(f(X)) \\ &= f\nabla_X Y - f\nabla_Y X - Y(f)X - fX(Y) + fY(X) + Y(f)X \\ &= f\left(\nabla_X Y - \nabla_Y X - [X, Y]\right) = fT(X, Y) \end{aligned}$$

And the same argument applies to $T(X, fY)$. ❖

As noted in [Dar94], the torsion has the following natural interpretation using the covariant exterior derivative (which links back to our consideration of the dual forms θ^i in terms of dI):
Consider $I \in \Omega^1(M, TM)$ given by $I(\eta) = \eta$; this makes sense as $I \colon TM \to TM$. We then have:

$$\begin{aligned} d_\nabla I(X, Y) &= d_\nabla I(Y)(X) - d_\nabla I(X)(Y) - I([X, Y]) \\ &= d_\nabla Y(X) - d_\nabla X(Y) - [X, Y] \\ &= \nabla_X Y - \nabla_Y X - [X, Y] = T(X, Y) \end{aligned}$$

For convenience, given a curve $\gamma \colon [a, b] \to M$, define the intrinsic derivative as $\frac{DX}{Dt} = \nabla_{\gamma_*(\frac{d}{dt})}X$, which generalises the usual notion of a derivative along a curve. We can then say that a vector field X is parallel along the curve γ iff $\frac{DX}{Dt} = 0$ along γ. We can now rephrase how a connection induces parallel translation, by considering a Koszul connection instead: given a curve γ and a vector $x \in T_{\gamma(a)}M$, the parallel translation of x along γ is the vector field X that is parallel along γ such that $X_{\gamma(a)} = x$.[31] This defines a map $\tau_t \colon T_{\gamma(a)}M \to T_{\gamma(t)}M$ which is an isomorphism of vector spaces (as it is

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[30] Following the comparison between ∇ and $\mathcal{L}$, we see that even though $\mathcal{L}_X Y$ and $\nabla_X Y - \nabla_Y X$ are non-tensorial, their difference is, which provides an interesting relation between the two different concepts of differentiation.

[31] The following explanation from [Spi99a] shows that this vector field is unique:
Consider the parallel vector field V along the curve γ, we can write $\frac{DV}{Dt} = \sum_{j=1}^n \frac{D}{Dt}\left(v_j \frac{\partial}{\partial x_j}\right) = \sum_{j=1}^n \left(\frac{dv_j}{dt}\frac{\partial}{\partial x_j} + v_j \nabla_{\frac{d\gamma}{dt}} \frac{\partial}{\partial x_j}\right)$ which is then $\frac{DV}{Dt} = \sum_{k=1}^n \left(\frac{dv_k}{dt} + \sum_{i,j=1}^n \Gamma_{ij}^k \frac{d\gamma}{dt} v_j\right)\frac{\partial}{\partial x_k}$. Then because V is parallel along γ we have that all the $\left(\frac{dv_k}{dt} + \sum_{i,j=1}^n \Gamma_{ij}^k \frac{d\gamma}{dt} v_j\right)$ are 0, and this condition amounts to solving some linear ordinary differential equations which have an unique solution.

bijective with inverse parallel translation along the curve in the opposite direction, and linear from the properties of ∇).

If our manifold also has a Riemannian metric, it would be nice if this parallel transport was an isometry.

Definition 3.38 A connection ∇ is compatible with a metric g if the parallel translations $T_{\gamma(a)}M \to T_{\gamma(t)}M$ are isometries for any smooth curve $\gamma : [a, b] \to M$ for all $t \in [a, b]$ with respect to the metric $g_{\gamma(t)}$.

Proposition 3.39 A connection ∇ is compatible with the metric g iff for any vectors fields X, Y along γ, $\frac{d}{dt}g(X, Y) = \left\langle \frac{DX}{Dt}, Y \right\rangle + \left\langle X, \frac{DY}{Dt} \right\rangle$

Proof. This is really just a restatement of the definition above. ❖

In particular, notice that this implies $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$ at any point p by considering a curve γ that satisfies $\frac{\partial \gamma}{\partial t}(0) = X$.

Theorem 3.40 (Fundamental Theorem of Riemannian Geometry)
On a Riemannian manifold (M, g) there is an unique torsion-free connection ∇ compatible with g.

Proof. Following [Spi99a], suppose ∇ is compatible with the metric. By Proposition 3.39, we get that

$$\frac{\partial g_{jk}}{\partial x_i} = \frac{\partial}{\partial x_i}\left\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \right\rangle = \left\langle \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \right\rangle + \left\langle \frac{\partial}{\partial x_j}, \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_k} \right\rangle$$

We can then apply the same argument by cyclically permuting i, j, k, and by using that $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i}$ (because ∇ is torsion free and $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$), we obtain

$$\sum_{l=1}^{n} \Gamma_{ij}^{l} g_{lk} = \left\langle \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \right\rangle = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x_j} + \frac{\partial g_{jk}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_k} \right)$$

so that

$$\Gamma_{ij}^{l} = \sum_{k=1}^{n} g^{kl} \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x_j} + \frac{\partial g_{jk}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_k} \right)$$

This uniquely determines the connection coefficients in terms of the metric, which proves the theorem. ❖

This connection is called the Levi-Civita connection of the Riemannian manifold. The connection coefficients are then called the Christoffel symbols (which are sometimes defined with the equation $\Gamma^l_{ij} = \sum_{k=1}^n g^{kl}\frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x_j} + \frac{\partial g_{jk}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_k}\right))$, and the tensor field $R(X,Y)$ is then called the Riemann curvature tensor.

Theorem 3.41 If the Riemann curvature tensor of an n-dimensional Riemannian manifold M is everywhere 0, then M is locally isometric to $\mathbb{R}^n$ with its usual metric.

Proof. We will only give a sketch proof. At a point p of M, we can write the "normal coordinates" which are given by locally orthonormal vector fields, for which the Christoffel symbols all vanish at p. The flatness of the Levi-Civita connection ($R = 0$) then allows us to parallel transport this coordinate system everywhere, while conserving the inner products as the Levi-Civita connection is (by definition) compatible with the metric. This means that we have, in essence, n globally defined orthonormal vector fields $\frac{\partial}{\partial x_i}$: so we are in $\mathbb{R}^n$. ❖

3.5 Geodesics

Given a Koszul connection, we know that we can consider $\frac{DX}{Dt} = \nabla_{\gamma_*(\frac{d}{dt})}X$. We can then define a special type of curves γ, called geodesics, which parallel transport their own tangent vector, so that $\frac{D}{Dt}\gamma_*(\frac{\partial}{\partial t}) = 0$.[32] Their interest lies in the fact that on a Riemannian manifold, the geodesics of the Levi-Civita connection are precisely the locally length minimising paths between points, where the length of a curve is understood to be the integral $\int_a^b \sqrt{\langle\gamma_*(\frac{\partial}{\partial t})(\tau), \gamma_*(\frac{\partial}{\partial t})(\tau)\rangle}d\tau = \int_a^b \|\frac{d\gamma}{dt}\|dt$, analogously to the usual definition for parametrised curves in $\mathbb{R}^n$.

Theorem 3.42 On a Riemannian manifold, the geodesics of the Levi-Civita connection are locally length-minimising paths.

Proof. This follows from a rather lengthy argument given in [Spi99b], which we will only outline here.

Instead of minimising the length L from a to b of a path γ, we minimise the energy $E = \frac{1}{2}\int_a^b \|\frac{d\gamma}{dt}\|^2 dt$. Using the Euler-Lagrange equations, we eventually find that γ is a critical point of E if and only if $\frac{d^2\gamma_k}{dt^2} + \sum_{i,j=1}^n \Gamma^k_{ij}\frac{d\gamma_i}{dt}\frac{d\gamma_j}{dt} = 0$, which are precisely the geodesic equations. Here the Γ^k_{ij} are defined by $\Gamma^k_{ij} = \sum_{l=1}^n g^{kl}\frac{1}{2}\left(\frac{\partial g_{il}}{\partial x_k} + \frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{ij}}{\partial x_l}\right)$, which we showed is an equivalent definition of the Christoffel symbols of the Levi-Civita connection. In fact, if γ is a critical point of E, γ must be parametrised proportionally to arclength. A little calculation then shows that the critical points γ of L satisfy

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[32] As before, we can write out a set of ordinary differential equations corresponding to the fact that a curve is a geodesic, the "geodesic equations" $\frac{d^2\gamma_k}{dt^2} + \sum_{i,j=1}^n \Gamma^k_{ij}\frac{d\gamma_i}{dt}\frac{d\gamma_j}{dt} = 0$ for all k, which demonstrates the uniqueness of geodesics through a given point with given tangent vector.

$\frac{d^2\gamma_k}{dt^2} + \sum_{i,j=1}^n \Gamma^k_{ij}\frac{d\gamma_i}{dt}\frac{d\gamma_j}{dt} - \frac{d\gamma_k}{dt}\frac{\frac{d^2 s}{d^2}}{\frac{ds}{dt}} = 0$ where s is the arclength function $s(t) = L_a^t$. But we had that $\frac{d^2 s}{dt^2} = 0$ so critical points of E are critical points of L, and critical points of L become critical points of E by reparametrising by arclength. This then suffices to show that locally, the critical points of L (which are geodesics) are minimums. ❖

We now want to consider how the geodesics of different connections on the same manifold relate. To begin, given two connections ∇ and $\overline{\nabla}$, we can form their difference $D(X,Y) = \overline{\nabla}_X(Y) - \nabla_X(Y)$.

Proposition 3.43 $D(X,Y)$ is a tensor field.

Proof. $D(fX,Y) = \overline{\nabla}_{fX}(Y) - \nabla_{fX}(Y) = f(\overline{\nabla}_X(Y) - \nabla_X(Y)) = fD(X,Y)$
$D(X,fY) = \overline{\nabla}_X(fY) - \nabla_X(fY) = f\overline{\nabla}_X(Y) - f\nabla_X(Y) + X(f)Y - X(f)Y = fD(X,Y)$ ❖

Now, if we write $S(X,Y) = \frac{1}{2}(D(X,Y) + D(Y,X))$ and $A(X,Y) = \frac{1}{2}(D(X,Y) - D(Y,X))$, we can see that if $\overline{T}$ and T are the torsion tensors of $\overline{\nabla}$ and ∇, $2A(X,Y) = \overline{\nabla}_X Y - \nabla_X Y - \overline{\nabla}_Y X + \nabla_Y X = \overline{T}(X,Y) + [X,Y] - T(X,Y) - [X,Y] = \overline{T}(X,Y) - T(X,Y)$. Therefore ∇ and $\overline{\nabla}$ have the same torsion if and only if $A = 0$.

Theorem 3.44 The following are equivalent:

$$\overline{\nabla} \text{ and } \nabla \text{ have the same geodesics (with the same parametrisations)} \tag{1}$$
$$D(X,X) = 0 \quad \text{for all } X \tag{2}$$
$$S = 0 \tag{3}$$

Proof. The following proof is given in [Spi99a]:
(1) ⇒ (2): Take $X_p \neq 0$ at $p \in M$, and consider the geodesic (for ∇ and $\overline{\nabla}$ $\gamma : \gamma(0) = p$ and $\gamma'(0) = X_p$. We know that we have a vector field X around p that agrees with $\frac{d\gamma}{dt}$. Hence $D(X_p, X_p) = \overline{\nabla}_{X_p}X - \nabla_{X_p}X = 0 - 0 = 0$ as X agrees with $\frac{d\gamma}{dt}$, and γ is a geodesic.
(2) ⇒ (1): Let γ be a geodesic for ∇ and let X be as above, so that X agrees with $\frac{d\gamma}{dt}$. Then $\overline{\nabla}_{X_p}X = D(X_p, X_p) + \nabla_{X_p}X = 0 + 0 = 0$ so that γ is also a geodesic for $\overline{\nabla}$.
(2) ⇔ (3): Obviously $D(X,X) = 0 \Leftrightarrow S(X,X) = 0$. This then implies $0 = S(X+Y, X+Y) = S(X,X) + S(Y,Y) + 2S(X,Y) = 2S(X,Y)$ so $S = 0 \Leftrightarrow D(X,X) = 0$. ❖

This shows that the geodesics determine the connection, up to torsion:

Corollary 3.45 Two connections with the same geodesics are equal if and only if they have equal torsion.
For every connection, there exists an unique connection with the same geodesics and with no torsion.

Proof. The first part is immediate from Theorem 3.44, seeing as $D = S + A$, if $S = 0$ then $D = \overline{T} - T$, so that the difference tensor is zero if and only if the connections have the same torsion.
For the second part, given $\overline{\nabla}$, we can define $\nabla_X(Y) = \overline{\nabla}_X(Y) + \frac{1}{2}\overline{T}(X,Y)$.[33] Then $D = \frac{1}{2}\overline{T}(X,Y)$ which is skew symmetric, so $S = 0$ and indeed the two connections have the same geodesics. But $T = \overline{T} - 2A = \overline{T} - D = 0$ so ∇ is indeed torsion free. ❖

For completeness, we also note the following theorem:

Theorem 3.46 The following are equivalent:

$$\overline{\nabla} \text{ and } \nabla \text{ have the same geodesics (with possibly different parametrisations)} \tag{1}$$

$$\text{For every } X \text{ there exists a } \lambda \text{ such that } D(X,X) = \lambda X \tag{2}$$

$$\text{There is a unique 1-form } \omega \text{ such that } S(X,Y) = \omega(X)Y + \omega(Y)X \tag{3}$$

Proof. See [Spi99a]. ❖

3.6 Moving frames

Another notion we can generalise to manifolds is the idea of moving frames on a manifold, corresponding to the cases of the Serret-Frenet frame and the Darboux frame.
A (linear) frame is simply an ordered basis for a vector space V. On a manifold, a moving frame is simply a choice of frame at every point on the manifold that varies smoothly. Given a chart of $U \subset M$, we have the usual moving frame which is simply given at each point by the tangent vectors $\frac{\partial}{\partial x_i}$, the ordered basis being $(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$. Therefore n linearly independent vector fields determine a moving frame so we will not make the distinction between the two. On a Riemannian manifold (M, g), we can also consider orthonormal moving frames which are frames where at each point the ordered basis is orthonormal. By Gram-Schmidt, it is possible to construct an orthonormal moving frame out of any moving frame.
Following [Spi99a], consider first a moving frame on $\mathbb{R}^n$ given by vector fields

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[33] One can check that this does indeed define a connection, for example by checking the properties of Proposition 3.30.

$(X_1, \dots, X_n)$. We want to measure quantitatively how this moving frame "moves". In this case, we can consider the vector fields as functions $X_i \colon \mathbb{R}^n \to \mathbb{R}^n$. Given the identity $I \colon \mathbb{R}^n \to \mathbb{R}^n$, we have that $dI(X) = X$, so we can define dual forms to the vector fields X_i by setting $dI = \sum_i \theta^i X_i$ or simply $dI = \theta X$ when considering θ and X as matrices of 1-forms and vector fields, respectively. We can also define ω by $dX_j = \sum_i \omega^i_j X_i$ or $dX = \omega X$. Therefore $\omega^i_j(X)$ is the X_i component of $dX_j(X)$ at each point. As [Spi99a] notices, because we know that $dX_j(X)$ is the directional derivative of X_j in the X direction, we can see ω^i_j as being a measure of how X_j rotates towards X_i as we go along a curve with tangent vector X.

Theorem 3.47 (Structure Equations in $\mathbb{R}^n$)

$$d\theta = -\omega \wedge \theta \tag{1}$$

$$d\omega = -\omega \wedge \omega \tag{2}$$

Proof. As in [Spi99a], write $dI = \sum_i \theta^i \wedge X_i$. Then:

$$\begin{aligned} 0 = d^2 I = d\Big(\sum_i \theta^i \wedge X_i\Big) &= \sum_i d\theta^i \wedge X_i - \sum_k \theta^k \wedge dX_k \\ &= \sum_i d\theta^i X_i - \sum_k \theta^k \wedge \left(\sum_i \omega^i_k X_i \right) \qquad \text{as } dX_j = \sum_i \omega^i_j X_i \end{aligned}$$

This proves (1). For (2):

$$\begin{aligned} 0 = d^2 X_j &= \sum_i d\omega^i_j X_i - \sum_k \omega^k_j \wedge dX_k \\ &= \sum_i d\omega^i_j X_i - \sum_k \omega^k_j \wedge \left(\sum_i \omega^i_k \wedge X_i \right) \end{aligned}$$

❖

On a general manifold, we can still define the dual forms θ^i by $\theta^i(X_j) = \delta^i_j$, but the expression dX doesn't make sense. The goal of that definition was to consider the directional derivatives, so using a Koszul connection on TM we define ω^i_j by $\nabla_{X_k}(X_j) = \sum_i \omega^i_j(X_k) X_i$ which we write as $\nabla(\mathbf{X}) = \mathbf{X} \cdot \omega$.
Suppose we have a second moving frame $\mathbf{X}' = \mathbf{X} \cdot a$ (which just means $X'_i = \sum_j a^j_i X_j$). We then have that $\mathbf{X}' \cdot \omega' = \nabla \mathbf{X}' = \nabla(\sum_l a^l_j X_l)$. As $\nabla_X(fY) = f\nabla_X Y + X(f)Y = f\nabla_X Y + df(X)Y$, we have that $\nabla(\sum_l a^l_j X_l) = \sum_l a^l_j \nabla(X_l) + \sum_l da^l_j X_l = \nabla\mathbf{X} \cdot a + \mathbf{X} \cdot da = \mathbf{X} \cdot (\omega \cdot a) + \mathbf{X} \cdot da$.

Therefore $\mathbf{X} \cdot (a\omega') = (X \cdot a)\omega' = X' \cdot \omega' = \mathbf{X} \cdot (\omega \cdot a) + \mathbf{X} \cdot da = \mathbf{X} \cdot (\omega \cdot a + da)$.
This leads us to the following definition:

Definition 3.48 A Cartan connection on a manifold is an assignment of a matrix $\omega = \omega^i_j$ of 1-forms to every moving frame $\mathbf{X}$ such that the matrix ω' corresponding to $\mathbf{X}' = \mathbf{X} \cdot a$ satisfies $\omega' = a^{-1}\omega a + a^{-1}da$.

We call the matrix ω the connection form corresponding to the connection, and the θ^i are the dual forms of the X_i.
In this more general case, the structure equations need adjustments, so we define Θ by $d\theta = -\omega \wedge \theta + \Theta$ and Ω by $d\omega = -\omega \wedge \omega + \Omega$.

Proposition 3.49 If Ω' and Θ' are the corresponding matrices for $\mathbf{X}' = \mathbf{X} \cdot a$, we have that:
$\Omega' = a^{-1}\Omega a$
$\Theta' = a^{-1} \cdot \Theta$

Proof. It suffices to compute this using the structure equations and the equation $\omega' = a^{-1}\omega a + a^{-1}da$. Details can be found in [Spi99a]. ❖

Theorem 3.50 Define $T(X_j, X_k) = \sum_i \Theta^i(X_j, X_k)X_i$ and $R(X_k, X_l)X_j = \sum_i \Omega^i_j(X_k, X_l)X_i$. Then T and R are the usual torsion and curvature tensors for the (Koszul) connection defined by $\nabla(\mathbf{X}) = \mathbf{X} \cdot \omega$.

Proof. Omitted, see [Spi99a]. ❖

As a consequence, we can notice that $\Omega^i_j(X_k, X_l) = \langle R(X_k, X_l)X_j, X_i\rangle$ and $\Theta^i(X_j, X_k) = \langle T(X_j, X_k), X_i\rangle$.

3.7 Principal bundles

The above idea of connection forms for a moving frame has the disadvantage of depending on a choice of moving frame on M. To avoid such dependence, we can think of moving frames as sections of the set of all frames on M. This leads us to the concept of principal bundles, as given in [Spi99a]:

Definition 3.51 A principal G-bundle on a manifold M consists of a (smooth) fiber bundle $\pi\colon E \to M$ and of a Lie group G with smooth (right) action $\cdot\colon E \times G \to E$, $\cdot\colon (u, g) \mapsto u \cdot g$ such that:
$\pi(u \cdot g) = \pi(u)$
$u \cdot (gh) = (u \cdot g) \cdot h$

For each $p \in M$ there exists a neighborhood $p \in U \subset M$ and a diffeomorphism $t\colon \pi^{-1}(U) \to U \times G$ of the form $t(u) = (\pi(u), \phi(u))$ such that $\phi(u \cdot g) = \phi(u)g$.

From these conditions, we can notice that $\{u \cdot g : g \in G\} = \pi^{-1}(\pi(u))$ (that is, the fiber in which u lies) as $\pi(u \cdot g) = \pi(u)$ and $\phi(u \cdot g) = \phi(u)g$. Also, we have that G acts freely and transitively on all such fibers,[34] so each fiber is diffeomorphic to G itself.

Coming back to moving frames, we can see that the set of all moving frames on a manifold M of dimension n (denoted $F(M)$) is a principal $GL(n, \mathbb{R})$-bundle, as the action of $GL(n, \mathbb{R})$ on each frame (at a point) is clearly transitive and free. Another example is the set of all orthonormal moving frames $OF(M)$ on M when M is equipped with a Riemannian metric, which is a principal $O(n, \mathbb{R})$-bundle.

We can also speak of oriented manifolds: we say a (C^∞) m dimensional manifold M is orientable iff there exists a differential m-form ω defined on M that is nowhere 0. This comes from the intuitive idea that on an orientable (connected) manifold, if we define a smooth choice of ordered basis for T_xM, we cannot get from a basis in one orientation to one in another.[35] If the manifold isn't orientable, we can move a positively oriented basis into a negatively oriented basis, and ω has to take opposite signs, so by the Intermediate Value Theorem $\omega = 0$ somewhere "in between". So if M is oriented, we can also consider the positively oriented orthonormal frames $SOF(M)$ which is a principal $SO(n, \mathbb{R})$-bundle.

We would now like to generalise the idea of a Cartan connection to this new framework. We can see that the definition of a Cartan connection can be rephrased as follows: A Cartan connection is an assignment of a $n \times n$ matrix-valued 1-form ω_s to every section $s\colon U \to F(M)$ such that $\omega_{s \cdot a} = a^{-1}da + a^{-1}\omega_s a$ for every smooth function $a\colon U \to GL(n, \mathbb{R})$. To generalise this notion, instead of considering all the ω_s corresponding to moving frames, we would like to have an $n \times n$ matrix-valued 1-form ω defined on the manifold $F(M)$ such that $\omega_s = s^*(\omega)$. We are then led to the consideration of $n \times n$ matrix-valued 1-forms ω on $F(M)$ such that $(s \cdot a)^*(\omega) = a^{-1}da + a^{-1}s^*(\omega)a$. As shown in [Spi99a], ω satisfies this transformation rule if and only if $\omega(\sigma(X)) = X$ and $\omega((R_A)_*(Y)) = \mathrm{Ad}(A^{-1})\omega(Y)$ where σ is the canonical map from tangent vectors to left invariant vector fields. This generalises to the following definition of a connection in a principal bundle.

Definition 3.52 A connection in a principal G-bundle $\pi\colon E \to M$ is a smooth $\mathfrak{g}$-valued

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[34] Which means that for all u, v in the same fiber, there exists an unique $g \in G$ such that $u = g \cdot v$ (the group action is transitive) and if $g \cdot u = u$ then $g = \mathrm{id}_G$ (the group action is free).

[35] Here, say $dx_1, dx_2, \ldots, dx_n$ is a positively oriented basis, and given any permutation $\sigma \in S_n$, the sign of the orientation of $dx_{\sigma(1)}, dx_{\sigma(2)}, \ldots, dx_{\sigma(n)}$ is $\mathrm{sign}(\sigma)$

1-form ω on E such that:
$\omega(\sigma(X)) = X$ for all $X \in \mathfrak{g}$
$\omega((R_g)_*(Y)) = \mathrm{Ad}(g^{-1})\omega(Y)$ for all $g \in G$ and $Y \in T_uE$ for any $u \in E$.

Given such a connection ω, we have a map $\omega\colon T_pE \to \mathfrak{g}$ at each point $p \in E$ which allows us to define an Ehresmann connection on E: the horizontal vectors are precisely those in the kernel $\ker(\omega)$.[36] This allows us to talk about horizontal and vertical components of tangent vectors.

Notice that the relation $\omega((R_g)_*(Y)) = \mathrm{Ad}(g^{-1})\omega(Y)$ for the connection ω shows that the connection form transforms "properly" under right translation. We can generalise this condition to arbitrary V-valued differential forms by requiring that, for a given representation $\rho\colon G \to GL(V)$, we have $(R_g)^*(\eta) = \rho(g^{-1})(\eta)$. We then say that η is pseudotensorial (of type (V, ρ)). The condition $\omega((R_g)_*(Y)) = \mathrm{Ad}(g^{-1})\omega(Y)$ for the connection form then just means that ω is pseudotensorial of type $(\mathfrak{g}, \mathrm{ad})$, where ad is the adjoint representation of $\mathfrak{g}$. A pseudotensorial form η is then called tensorial if it is horizontal, so that $\eta(X_1, \ldots, X_n) = 0$ if any of the X_i is vertical.

In the same way as the Koszul connection generalises to a covariant exterior derivative on differential forms, a connection in a principal bundles induces a covariant exterior derivative:

Define the covariant exterior derivative D of a pseudotensorial form η by $D\eta(X_1, \ldots, X_n) = (d\eta)(h(X_1), \ldots, h(X_n))$ where $h(X)$ is the horizontal component of X as given by the Ehresmann connection associated to the principal bundle connection. In particular, this means that $D\eta$ is tensorial.

This covariant exterior derivative D allows us to define the curvature of the connection ω by $\Omega = D\omega$.[37] We might also want to define the dual forms θ and give a similar definition of the torsion, but we don't seem to have anything to work with as we don't have any vector fields given. In fact, it happens that it is only possible to define the torsion of a connection when the principal bundle is a bundle of frames, because this allows us to consider the 1-form θ given by $\theta_g(X) = g^{-1}(\pi_* X)$ for any tangent vector $X \in T_{(x,g)}F(M)$. This definition might seem mysterious, but is explained by the fact that if we take any section s of $F(M)$ given by $s = (X_1, \ldots, X_n)$, we find that $s^*(\theta(Y)) = \theta_{s(p)}(s_* Y) = (s(p))^{-1}(Y)$ so that the θ^i are just the dual forms for the moving frame $(X_1, \ldots, X_n)$. We can then define the torsion form of a connection ω on

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[36] For a proof that this does indeed provide us with a well-defined Ehresmann connection, see, for example, [Spi99a]. We also note that an Ehresmann that is "compatible" with the group structure (such that $H_{u\cdot g} = (R_g)_* H_u$) of the principal bundle induces a principal bundle connection.

[37] We will show that this definition corresponds to the old definition of curvature form for a moving frame in Theorem 3.54.

a frame bundle by $\Theta = D\theta$.

Notice that for $\omega \in \Omega^1(M, \mathfrak{g})$ we have $[\omega, \omega](X, Y) = \frac{1}{2}[\omega(X), \omega(Y)]$ as $[\omega, \omega](X, Y) = \frac{1}{2}([\omega(X), \omega(Y)] - [\omega(Y), \omega(X)]) = \frac{1}{2}[\omega(X), \omega(Y)]$.

For tensorial forms of type (V, ρ), the appearance of the covariant exterior derivative is in fact quite simple:

Proposition 3.53 $D\alpha = d\alpha + \rho(\omega) \wedge \alpha$.

Proof. Omitted. It suffices to carry out the calculation with various combinations of horizontal and vertical vectors, using the known properties of the covariant exterior derivative as well as the expression given in Remark 3.13. ❖

In the case that ρ is the adjoint representation, we have that $D\alpha = d\alpha + [\omega, \alpha]$, which follows from Proposition 3.21.

But ω and θ are not tensorial as they are not horizontal, and we could instead expect that the structure equations hold, so that $D\omega = \Omega = d\omega + \frac{1}{2}[\omega, \omega]$, and similarly for Θ. This is indeed the case:

Theorem 3.54 (Structure Equations)

$$d\theta = -\frac{1}{2}[\omega, \theta] + \Theta \tag{1}$$

$$d\omega = -\frac{1}{2}[\omega, \omega] + \Omega \tag{2}$$

Proof. We just need to compute $d\theta$ and $d\omega$ on various combinations of horizontal and vertical vectors, and the result falls out easily. Refer to [Spi99a] for a written-out calculation. ❖

Notice that this theorem shows that the definitions $\Omega = D\omega$ and $\Theta = D\theta$ of the curvature and torsion forms do indeed coincide with the definitions we made in the case of moving frames. We also have the following identities that we had before:

Proposition 3.55

$$R_g^*(\Omega) = \mathrm{Ad}(g^{-1})(\Omega)$$
$$R_g^*(\Theta) = g^{-1}(\Theta)$$

Proof. Omitted, see [Spi99a]. ❖

Theorem 3.56 (Bianchi Identities)

$$D\Theta = \Omega \wedge \theta \tag{1}$$

$$D\Omega = 0 \tag{2}$$

Proof. (1) From Theorem 3.54 (1), we have that $d\theta = -\frac{1}{2}[\omega, \theta] + \Theta$. Therefore $0 = -d\omega \wedge \theta + \omega \wedge d\theta + d\Theta$, hence $D\Theta(X, Y, Z) = d\Theta(h(X), h(Y), h(Z)) = (d\omega \wedge \theta)(h(X), h(Y), h(Z)) - 0 = (\Omega \wedge \theta)(X, Y, Z)$ as $d\omega(h(X), h(Y)) = \Omega(X, Y)$, $\theta(h(X)) = \theta(X)$ and ω is horizontal.
(2) From Theorem 3.54 (2), we have that $d\omega = -\frac{1}{2}[\omega, \omega] + \Omega$. Therefore $0 = -d\omega \wedge \omega + \omega \wedge d\omega + d\Omega$. Then $D\Omega(X, Y, Z) = d\omega(h(X), h(Y), h(Z)) = (d\omega \wedge \omega)(h(X), h(Y), h(Z)) - (\omega \wedge d\omega)(h(X), h(Y), h(Z)) = 0$ as ω is horizontal (as [Spi99a] notices, $d\omega \wedge \omega$ and $\omega \wedge d\omega$ do not cancel out as matrix multiplication is not usually commutative). ❖

We can also recover the fact that $D \circ D$ is directly related to the curvature of the connection, as follows.

Proposition 3.57 For tensorial forms η, we have that $(D \circ D)(\eta) = [\Omega, \eta]$

Proof. We can write $D\eta = d\eta + [\omega, \eta]$ as η is tensorial. Then $D^2\eta = [d\omega \wedge \eta + \frac{1}{2}[\omega, \omega], \eta] = [\Omega, \eta]$ by the Second Structure Equation. ❖

3.8 Holonomy

Given a Koszul connection ∇ on a vector bundle, we might wonder how the parallel transport it induces fails to preserve structure: it might be the case that a tangent vector is transported along a loop from a point back to itself to a different tangent vector. Every loop $\gamma \colon [a, b] \to M$ with $\gamma(a) = \gamma(b) = p$ induces a parallel transport $\tau_\gamma \colon T_pM \to T_pM$ dependent on the connection ∇. The holonomy group is then defined as $\mathrm{Hol}(\nabla, p) = \{\tau_\gamma\}$, which has a natural group structure under composition of loops. We can also define $\mathrm{Hol}_0(\nabla, p)$ to be the subgroup of $\mathrm{Hol}(\nabla, p)$ where we only consider parallel transports along loops which are homotopic to the constant loop. We can similarly define the holonomy group for a connection on a principal bundle: $\mathrm{Hol}(\omega, p) = \{g \in G : \text{there exists a (smooth) horizontal path connecting } g \text{ and } p \cdot g\}$ where "horizontal" is dependent on ω.

Proposition 3.58

$$\mathrm{Hol}(\omega, p\cdot g) = \mathrm{Ad}(g^{-1})\mathrm{Hol}(\omega, p) \tag{1}$$

$$\mathrm{Hol}_0(\omega, p) = 0 \iff \Omega = 0 \tag{2}$$

$$\text{There is a surjective homomorphism } \pi_1(M) \to \mathrm{Hol}(\omega,p)/\mathrm{Hol}_0(\omega,p) \tag{3}$$

Proof. (1) is clear from the fact that to get from Hol at p to Hol at $p\cdot g$, one just needs to add the path p to $p\cdot g$ and back.

(2)($\Leftarrow$) This essentially follows from the proof of 3.41, as we realised that if a connection is flat, then we can write linearly independent vector fields for which the Christoffel symbols vanish. This then implies triviality of Hol_0.

($\Rightarrow$) This can be shown from considering small loops corresponding to $\nabla_X\nabla_Y - \nabla_Y\nabla_X$, and seeing that this is always 0. We then recover the statement for Ω by the correspondence we had established between the two notions of curvature.

(3) Hol_0 is clearly a normal subgroup of Hol as if $\gamma \in \mathrm{Hol}_0$ then for $c \in \mathrm{Hol}$ we have that $c^{-1}\gamma c$ is homotopic to γ. Given any loop c based at p, we get a corresponding parallel transport $\tau_{c^{-1}}$, and if two loops c and c' are homotopic, then $\tau_{c'} \circ \tau_{c^{-1}} = \tau_{c^{-1}c'} \in \mathrm{Hol}_0$. Hence any two homotopic loops get mapped to the same element in $\mathrm{Hol}/\mathrm{Hol}_0$ so we can consider it as a map $\pi_1 \to \mathrm{Hol}/\mathrm{Hol}_0$. As every parallel transport in Hol arises from a loop, this shows that it is surjective. ❖

3.9 Generalised Gauss-Bonnet Theorem

We can generalise the notion of orientation to any vector bundle. Suppose ξ given by $\pi\colon E \to M$ is an n-dimensional vector bundle. An orientation of ξ is an orientation of all fibers $\pi^{-1}(p)$ for $p \in M$, such that the linear isomorphisms φ in the definition of a vector bundle preserve the orientation on the fibers. Then the usual definition of orientation of a manifold corresponds to the orientation of the tangent bundle of M.

An orientation μ of a k-dimensional vector bundle ξ allows us to determine an element $U \in H^k_c(E)$ called the Thom class by stipulating that for all $p \in M$, $i_p^*(\omega) = \mu$ where i_p is just the inclusion $\pi^{-1}(p) \to E$, and ω is a closed form representing U.

This now allows us to find a class $\chi(\xi) \in H^k(M)$ called the Euler class by setting $\chi(\xi) = s^*(U)$ for any section s of ξ.[38] Note that if there exists a nowhere-zero section s of ξ, then $\chi(\xi) = 0$ as we can just multiply s by a constant c to get arbitrarily far away

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[38] As noted in [Spi99b], there is always one section (the 0-section), and any two sections are smoothly homotopic, so that χ is well-defined and independent of the choice of section.

from the origin of the vector space in each fiber. As U is compactly supported, we can eventually get out of the support of U and so $\chi(\xi) = (cs)^*(U) = 0$.

Given a vector bundle morphism $f\colon E \to F$ $g\colon M \to N$ of ξ_1: $\pi_1\colon E \to M$ and ξ_2 $\pi_2\colon F \to N$, χ is "natural": $f^*(\chi(\xi_2)) = \chi(\xi_1)$, and is compatible with Whitney sums, so that $\chi(\xi_1 \oplus \xi_2) = \chi(\xi_1) \wedge \chi(\xi_2)$.[39]

The name "Euler class" relies on the following theorem:

Theorem 3.59 If $\xi = \pi\colon TM \to M$ is the tangent bundle of M, then $\chi(\xi) = \chi(M)\mu$.

Proof. Omitted, see for example [Spi99b]. We can consider the sections of TM as vector fields on M, and the proof that $\chi(\xi) = \chi(M)\mu$ is a consequence of the Poincaré-Hopf theorem, which states that the sum of the indices[40] of the isolated zeros of a vector field is equal to the Euler characteristic of the manifold. This allows us to relate the Euler class of TM with the Euler characteristic of M. ❖

Suppose we are given a $2n$-dimensional vector bundle $\pi\colon E \to M$, and consider the bundle of positively oriented frames $\varpi\colon SOF(E) \to M$. The connection form ω defined on the bundle $SOF(E)$ then has values in the Lie algebra $\mathfrak{so}(2n)$ of even-dimensional skew-symmetric matrices by Proposition 3.20.

An interesting fact in linear algebra tells us that for skew-symmetric matrices A, the determinant $\det(A)$ can be written as the square of a polynomial in the matrix entries, $\mathrm{Pf}(A)$, the Pfaffian of A, so that $(\mathrm{Pf}(A))^2 = \det(A)$. This is only interesting if A is even dimensional, as odd-dimensional skew-symmetric matrices have determinant 0. If A is of dimension $2n$, $\mathrm{Pf}(A)$ is given by $\mathrm{Pf}(A) = \frac{1}{2^n n!}\sum_{\sigma\in S_{2n}} \mathrm{sign}(\sigma)\prod_{i=1}^{n} a_{\sigma(2i-1)}a_{\sigma(2i)}$. The factor $\frac{1}{2^n n!}$ can be explained by the fact that there is a lot of redundancy in this expression, as interchanging the pairs $(2i-1, 2i)$ and $(2j-1, 2j)$ does not change the expression, and interchanging the two elements in each pair doesn't either as the matrix is skew-symmetric. Therefore we are counting the same thing $2^n n!$ times, where the 2^n accounts for the n pairs whose elements can be switched, and the $n!$ corresponds to all the permutations of pairs possible. Therefore we could also write $\mathrm{Pf}(A) = \sum_{P\in\mathcal{P}} \mathrm{sign}(P)a_P$ where $\mathcal{P}$ is the set of sets of pairs $\{(j_1, k_1), \ldots, (j_n, k_n)\}$ with $j_i < k_i$ for all i, and $\mathrm{sign}(P)$ is the sign of the permutation corresponding to the set of pairs P. The importance of

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[39] For a proof of these facts, refer to [Spi75b]. In fact, these properties are sometimes taken as the defining properties of the Euler class, and it can be shown that the Euler class is the unique cohomology class that satisfies these three properties and Theorem 3.59 (or some other normalisation property).

[40] The index of a vector field on a plane at an isolated zero corresponds to the winding number of the vector field along a small loop around the zero. This can then be generalised to higher dimensional vector fields: say V has an isolated zero at $0 \subset U \subset \mathbb{R}^n$ with no other zeros in U, the index is then the degree of the map $f\colon (U \setminus 0) \to S^{n-1}$ given by $f(p) = X_p/\|X_p\|$, this is really the "obvious" way to generalise the winding number, but for higher dimensional spheres instead of circles.

the Pfaffian is given by the fact that $\mathrm{Pf}(B^tAB) = \det(B)\mathrm{Pf}(A)$ for any B and skew symmetric A[41], which ensures that the Pfaffian is invariant under multiplication by elements of $SO(n)$.

Another useful property of the Pfaffian is that $\mathrm{Pf}(A \oplus B) = \mathrm{Pf}(A)\mathrm{Pf}(B)$ where A and B are skew symmetric matrices.[42]

We can also define the Pfaffian for $\mathfrak{so}(n)$-valued differential forms, all we need to do is replace the product with the wedge product. In particular, we can consider $\mathrm{Pf}(\Omega)$ for a curvature form Ω, given by $\mathrm{Pf}(\Omega) = \frac{1}{2^n n!}\sum_{\sigma\in S_n}\mathrm{sign}(\sigma)\bigwedge_{j=1}^{n}\Omega^{\sigma(2j-1)}_{\sigma(2j)}$, which is indeed well defined as in a different orthonormal frame we have $\Omega' = A^{-1}\omega A$ so $\mathrm{Pf}(\Omega') = \mathrm{Pf}(A^{-1}\Omega A) = \mathrm{Pf}(A^t\Omega A) = \det(A)\mathrm{Pf}(\Omega) = \mathrm{Pf}(\Omega)$ as $A \in SO(n)$.

We are now ready to state the generalised Gauss-Bonnet Theorem:

Theorem 3.60 (Gauss-Bonnet-Chern)

Let M be a $m = 2n$ dimensional compact oriented Riemannian manifold (without boundary), with curvature form Ω of the set $SOF(TM)$ of orthonormal moving frames on TM. Then:

$$\int_M \mathrm{Pf}(\Omega) = (-2\pi)^n\,\chi(M)$$

Proof. Write $\mathrm{e}(\xi) = \left[\left(\frac{-1}{2\pi}\right)^n \mathrm{Pf}(\Omega)\right]$.

It can be shown that e shares many properties with χ that we saw earlier: it satisfies the Whitney sum formula $\mathrm{e}(\xi_1 \oplus \xi_2) = \mathrm{e}(\xi_1) \wedge \mathrm{e}(\xi_2)$ and is natural with fiber bundle morphisms.[43]

We then want to show that $\int_M \mathrm{e}(\xi) = \chi(M)$, which is the same as $\int_M \mathrm{e}(\xi) = \int_M \chi(\xi)$ by Theorem 3.59. Hence it suffices to show $\mathrm{e} = \chi$. Following Bell [Bel06], we are going to construct the Thom class explicitly for the case when the vector bundle TM has rank 2, and then proceed by induction using the relations $\mathrm{e}(E_1 \oplus E_2) = \mathrm{e}(E_1) \wedge \mathrm{e}(E_2)$ and $\chi(E_1 \oplus E_2) = \chi(E_1) \wedge \chi(E_2)$.

Given a positively oriented orthonormal moving frame (X_1, X_2) of the bundle, we know that $\Omega = d\omega + \omega \wedge \omega$. Then $\mathrm{Pf}(\Omega) = \Omega^1_2 = d\omega^1_2 + \omega^1_1 \wedge \omega^1_2 + \omega^1_2 \wedge \omega^2_2 = d\omega^1_2$ as $\omega^1_1 = \omega^2_2 = 0$ seeing as we are working with the Levi-Civita connection that is torsion-free.

We want to find that the Thom class satisfies $s^*(U) = \frac{-1}{2\pi}d\omega^2_1$ for some section s, which we

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[41] For a proof of this claim, see [Spi75b].

[42] This follows as $(\mathrm{Pf}(A \oplus B))^2 = \det(A \oplus B) = \det(A)\det B = (\mathrm{Pf}(A)\mathrm{Pf}(B))^2$. Then $\mathrm{Pf}(A \oplus B) = \pm\mathrm{Pf}(A)\mathrm{Pf}(B)$, and it must be + as the sign must be constant for all A, B (as Pf is a continuous function of the matrix entries, and we can move from any A to any B while keeping Pf nonzero), and in the case when A and B are of the form $\bigoplus_{i=1}^{k}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, we get the required + sign.

[43] Refer to [Spi75b] for a proof of these properties.

can consider to be the 0 section. So we start by guessing $U = f\pi^*(d\omega_2^1)$ for some smooth function f on $E = TM$. But then $dU = df \wedge \pi^*(d\omega_2^1)$ but we want $dU = 0$. Therefore consider $U = f\pi^*(d\omega_2^1) + df \wedge \pi^*(\omega_2^1)$, $dU = 0$. To make things easier, suppose f is a function of the distance to the origin r in each fiber and consider the local coordinate system (r, θ) given by the polar coordinates in any local trivialisation. We want f constant near $r = 0$ so that df pulls back to 0 under the 0 section.

We realise that our expression for U is not invariant under an orthonormal change of coordinates, as we get that given $A = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix}$,

$$A^{-1}dA = \begin{pmatrix} -\sin(\varphi)\cos(\varphi)d\varphi + \sin(\varphi)\cos(\varphi)d\varphi & -(\cos(\varphi))^2 d\varphi - (\sin(\varphi))^2 d\varphi \\ (\sin(\varphi))^2 d\varphi + (\cos(\varphi))^2 d\varphi & \sin(\varphi)\cos(\varphi)d\varphi - \cos(\varphi)\sin(\varphi) \end{pmatrix}$$

which is just $\begin{pmatrix} 0 & -d\varphi \\ d\varphi & 0 \end{pmatrix}$ so $(A^{-1}dA)_2^1 = -d\varphi$ and $(A^{-1}\omega A)_2^1 = -\omega_1^1 \cos(\varphi)\sin(\varphi) + \omega_1^2(\cos(\varphi))^2 + \omega_2^1(\sin(\varphi))^2 + \omega_2^2 \sin(\varphi)\cos(\varphi) = \omega_1^2$ as $\omega_1^1 = \omega_2^2 = 0$, $\omega_1^2 = \omega_2^1$. Hence $\omega'^1_2 = (A^{-1}dA)_2^1 + (A^{-1}\omega A)_2^1 = -d\varphi + \omega_2^1$.

Note that $d\theta' = d\theta + d\varphi$, so consider $U = f\pi^*(d\omega_2^1) + df \wedge \pi^*(\omega_2^1) + df \wedge \pi^*(d\theta)$. This is then correctly invariant under orthonormal transformations as can be seen from the above calculation of ω'^2_1. We also still have $dU = 0$, so we just need to see what it pullbacks to under the 0 section. We get $s^*(U) = f(0)d\omega_2^1$ so we put $f(0) = (-2\pi)^{-1}$. We now have to check that the integral over each fiber of the Thom class is 1. If i is the inclusion $\pi^{-1}(p) \to E$, the integral is then given by $\int_{\pi^{-1}(p)} i^*(U) = \int_0^{+\infty}\int_0^{2\pi} d(i^*(f)) \wedge i^*\pi^*(d\theta) = \int_0^{+\infty} df \int_0^{2\pi} d\theta = -f(0) \cdot 2\pi = 1$ as $f(0) = \frac{-1}{2\pi}$ and $\lim_{r\to\infty} f(r) = 0$ because df is compactly supported, which is what we wanted.[44] Therefore the theorem is true for vector bundles of rank 2.

Now in the case that $E = E_1 \oplus \cdots \oplus E_k$, we can write $\mathrm{e}(\xi) = (-2\pi)^{-n}\mathrm{Pf}(\Omega) = \mathrm{e}(\xi_1) \wedge \cdots \wedge \mathrm{e}(\xi_k)$ as $\mathrm{Pf}(A \oplus B) = \mathrm{Pf}(A)\mathrm{Pf}(B)$. Then $\mathrm{e}(\xi_1) \wedge \cdots \wedge \mathrm{e}(\xi_n) = \chi(\xi_1) \wedge \cdots \wedge \chi(\xi_k)$ by the previous part of the proof. As $\chi(\xi_1) \wedge \chi(\xi_2) = \chi(\xi_1 \oplus \xi_2)$ we have that $\chi(\xi_1) \wedge \cdots \wedge \chi(\xi_k) = \chi(\xi_1 \oplus \cdots \oplus \xi_k)$. Therefore $\mathrm{e} = \chi$ in the case when we can decompose E in a direct sum of rank 2 bundles.

The following Lemma allows us to finish the proof:

Lemma 3.61 (Splitting Principle)

If $\pi\colon E \to M$ is an even dimensional orientable vector bundle, there exists a manifold N

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[44] This also shows why the constant in front of the Pfaffian for e is what it is, as the choice of $f(0)$ was forced upon us and it turns out it is the correct one to define the Thom class. If we had chosen any other constant, we couldn't have got the two equations to hold.

and a smooth map $g\colon N \to M$ such that $g^*\colon H^\bullet(M) \to H^\bullet(N)$ is injective and $g^*(E)$ is a direct sum of orientable rank 2 vector bundles.

Indeed, we then have that $g^*(\mathrm{e}(\xi)) = \mathrm{e}(g^*(\xi)) = \chi(g^*(\xi)) = g^*(\chi(\xi))$, and because g^* is injective we have that $\mathrm{e} = \chi$. Hence the theorem holds.[45] ❖

In two dimensions, we have that $\Omega^i_j(X_k, X_l) = \langle R(X_k, X_l, X_j)X_i\rangle$. But we know that $K = \frac{\det(II(X_i,X_j))}{\det(I(X_i,X_j))} = \langle R(X_i, X_j, X_j)X_i\rangle$. Hence $\mathrm{Pf}(\Omega) = \Omega^1_2 = K d\theta^2 \wedge d\theta^1 = -K dA$ where θ^i is the dual 1-form to X_i, so we recover $\int_M K dA = 2\pi\chi(M)$.

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[45] Notice that the theorem actually holds in a slightly more general setting: we can consider a general vector bundle E of rank $2n$ over M, not just TM, provided we have a sufficient generalisation of Theorem 3.59. In that case, instead of the Levi-Civita connection (which requires $E = TM$), we can just use any connection compatible with the metric.

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